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Approximation by Multivariate Singular Integrals

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Approximation by Multivariate Singular Integrals

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To my wife Koula and my daughters
Angela and Peggy

Preface

This short monograph is the first to deal exclusively with the study of the approximation of multivariate singular integrals to the identity-unit operator. Here we study quantitatively the basic approximation properties of the general multivariate singular integral operators, special cases of which are the multivariate Picard, Gauss-Weierstrass, Poisson-Cauchy and trigonometric singular integral operators, etc. These operators are not general positive linear operators. In particular we study the rate of convergence of these operators to the unit operator, as well as the related simultaneous approximation. These are given via inequalities and by the use of multivariate higher order modulus of smoothness of the high order partial derivatives of the involved function. Also we study the global smoothness preservation properties of these operators. Some of these multivariate inequalities are proved to be attained, that is sharp. Furthermore we give asymptotic expansions of Voronovskaya type for the error of approximation. These properties are studied with respect to Lp norm, $1 \leq p \leq \infty$. The last chapter presents a related Korovkin type approximation theorem for functions of two variables. Plenty of examples are given.

For the convenience of the reader, the chapters are self-contained.

This brief monograph relies on author's last two years of related research work, more precisely see author's articles in the list of references of each chapter.

Advanced courses can be taught out of this short book. All necessary background and motivations are given per chapter.

The presented results are expected to find applications in many areas of pure and applied mathematics, such as mathematical analysis, probability, statistics and partial differential equations, etc. As such this brief monograph is suitable for researchers, graduate students, and seminars of the above subjects, also to be in all science libraries.

The preparation of this book took place during 2010–2011 in Memphis, Tennessee, USA.

I would like to thank my family for their dedication and love to me, which was the strongest support during the writing of thismonograph.

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Contents

Chapter 1
Uniform Approximation by General Multivariate Singular Integral Operators

In this chapter, we present the uniform approximation properties of general multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$. We give their convergence to the unit operator with rates. The estimates are pointwise and uniform. The established inequalities involve the multivariate higher order modulus of smoothness. We list the multivariate Picard, Gauss-Weierstrass, Poisson Cauchy and trigonometric singular integral operators where this theory can be applied directly. This chapter relies on [2].

1.1 Introduction

The rate of convergence of univariate singular integral operators has been studied in [1, 7–9, 11, 12], and these works motivate the current chapter. Here we consider some very general multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$, and we study the degree of approximation to the unit operator with rates over smooth functions. We present related inequalities involving the multivariate higher modulus of smoothness with respect to $\|\cdot\|_\infty$. The estimates are pointwise and uniform. See Theorems 1.9, 1.11. We mention particular operators that fulfill our theory. The discussed linear operators are not in general positive. Other motivation comes from [4, 5].

1.2 Main Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \ldots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-m}, & \text{if } j = 0, \end{cases} \tag{1.1}$$

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and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \ldots, m \in \mathbb{N}. \tag{1.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{1.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{1.4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]} (f; x_1, \ldots, x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f\left(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j\right) \mathrm{d}\mu_{\xi_n}(s), \tag{1.5}$$

where $s := (s_1, \ldots, s_N)$, $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

Remark 1.1. The operators $\theta_{r,n}^{[m]}$ are not in general positive. For example, consider the function $\varphi(u_1, \ldots, u_N) = \sum_{i=1}^{N} u_i^2$ and also take $r = 2$, $m = 3$; $x_i = 0$, $i = 1, \ldots, N$. See that $\varphi \geq 0$, however

$$\begin{aligned}
\theta_{2,n}^{[3]} (\varphi; 0, 0, \ldots, 0) &= \left(\sum_{j=1}^{2} j^2 \alpha_{j,2}^{[3]}\right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2\right) \mathrm{d}\mu_{\xi_n}(s) \\
&= \left(\alpha_{1,2}^{[3]} + 4\alpha_{2,2}^{[3]}\right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2\right) \mathrm{d}\mu_{\xi_n}(s) \\
&= \left(-2 + \frac{1}{2}\right) \int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2\right) \mathrm{d}\mu_{\xi_n}(s) < 0,
\end{aligned} \tag{1.6}$$

assuming that $\int_{\mathbb{R}^N} \left(\sum_{i=1}^{N} s_i^2\right) \mathrm{d}\mu_{\xi_n}(s) < \infty$.

Lemma 1.2. *The operators $\theta_{r,n}^{[m]}$ preserve the constant functions in N variables.*

Proof. Let $f(x_1, \ldots, x_N) = c$, then

$$\theta_{r,n}^{[m]}(c; x_1, \ldots, x_n) = \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^n} c\, d\mu_{\xi_n}(s) = c.$$

□

We need

Definition 1.3. Let $f \in C_B(\mathbb{R}^N)$, the space of all bounded and continuous functions on $\mathbb{R}^N$. Then, the rth multivariate modulus of smoothness of f is given by (see, e.g. [6])

$$\omega_r(f; h) := \sup_{\sqrt{u_1^2 + \ldots + u_N^2} \le h} \left\| \Delta_{u_1, u_2, \ldots, u_N}^{r}(f) \right\|_\infty < \infty, \quad h > 0, \tag{1.7}$$

where $\|\cdot\|_\infty$ is the sup-norm and

$$\Delta_u^r f(x) := \Delta_{u_1, u_2, \ldots, u_N}^{r} f(x_1, \ldots, x_N)$$
$$= \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x_1 + j u_1, x_2 + j u_2, \ldots, x_N + j u_N). \tag{1.8}$$

Let $m \in \mathbb{N}$ and let $f \in C^m(\mathbb{R}^N)$.

Suppose that all partial derivatives of f of order m are bounded, i.e.

$$\left\| \frac{\partial^m f(\cdot, \cdot, \ldots, \cdot)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \right\|_\infty < \infty, \tag{1.9}$$

for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N$; $\sum_{j=1}^{N} \alpha_j = m$.

We make

Remark 1.4. Let $l = 0, 1, \ldots, m$. The lth order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$ and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$.

Consider $g_z(t) := f(x_0 + t(z - x_0))$, $t \ge 0$; $x_0, z \in \mathbb{R}^N$.

Then

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t(z_1 - x_{01}), \ldots, x_{0N} + t(z_N - x_{0N})), \tag{1.10}$$

for all $j = 0, 1, \ldots, m$.

We have the multivariate Taylor's formula, [3]:

$$f(z_1, \ldots, z_N) = g_z(1) = \sum_{j=0}^{m} \frac{g_z^{(j)}(0)}{j!} + \frac{1}{(m-1)!} \int_0^1 (1-\theta)^{m-1} \left(g_z^{(m)}(\theta) - g_z^{(m)}(0) \right) d\theta. \tag{1.11}$$

Notice $g_z(0) = f(x_0)$. Also for $j = 0, 1, \ldots, m$, we have

$$g_z^{(j)}(0) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N} \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0). \tag{1.12}$$

Furthermore,

$$g_z^{(m)}(\theta) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N} \alpha_i = m}} \left(\frac{m!}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (z_i - x_{0i})^{\alpha_i} \right) f_\alpha(x_0 + \theta(z - x_0)), \tag{1.13}$$

$0 \leq \theta \leq 1$.

We apply the above for

$$z = (z_1, \ldots, z_N) = (x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) = x + sj,$$

and

$$x_0 = (x_{01}, \ldots, x_{0N}) = (x_1, x_2, \ldots, x_N) = x,$$

to obtain

$$f(x_1 + s_1 j, \ldots, x_N + s_N j) = g_{x+sj}(1) = \sum_{\tilde{j}=0}^{m} \frac{g_{x+sj}^{(\tilde{j})}(0)}{\tilde{j}!} + \frac{1}{(m-1)!} \times \int_0^1 (1-\theta)^{m-1} \left(g_{x+sj}^{(m)}(\theta) - g_{x+sj}^{(m)}(0) \right) d\theta, \tag{1.14}$$

where $g_{x+sj}(t) := f(x + t(sj))$.

Notice $g_{x+sj}(0) = f(x)$.

Also for $\tilde{j} = 0, 1, \ldots, m$ we have

$$g_{x+sj}^{(\tilde{j})}(0) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=\tilde{j}}} \left(\frac{\tilde{j}!}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (s_i j)^{\alpha_i} \right) f_\alpha(x). \tag{1.15}$$

Furthermore, we get

$$g_{x+sj}^{(m)}(\theta)/m! = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (s_i j)^{\alpha_i} \right) f_\alpha\left(x + \theta\left(sj\right)\right), \tag{1.16}$$

$0 \leq \theta \leq 1$.

For $\tilde{j} = 1, \ldots, m$ and $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \tilde{j}$, will be proved that

$$c_{\alpha,n,\tilde{j}} := c_{\alpha,n} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, d\mu_{\xi_n}(s_1, \ldots, s_N) \in \mathbb{R}, \tag{1.17}$$

see (1.35).

Consequently, we derive

$$\sum_{\tilde{j}=1}^{m} \frac{\int_{\mathbb{R}^N} g_{x+sj}^{(\tilde{j})}(0) d\mu_{\xi_n}(s)}{\tilde{j}!}$$

$$= \sum_{\tilde{j}=1}^{m} j^{\tilde{j}} \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i=\tilde{j}}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) c_{\alpha,n} f_\alpha(x) \right). \tag{1.18}$$

Next we observe that

$$\frac{1}{(m-1)!}\int_{\mathbb{R}^N}\left(\int_0^1 (1-\theta)^{m-1}\left(g_{x+sj}^{(m)}(\theta)-g_{x+sj}^{(m)}(0)\right)\mathrm{d}\theta\right)\mathrm{d}\mu_{\xi_n}(s)$$

$$= mj^m \sum_{\left(\substack{\alpha:=(\alpha_1,\dots,\alpha_N),\ \alpha_i\in\mathbb{Z}^+,\\ i=1,\dots,N,\ |\alpha|:=\sum_{i=1}^N \alpha_i=m}\right)} \frac{1}{\left(\prod_{i=1}^N \alpha_i!\right)}\cdot$$

$$\left(\int_{\mathbb{R}^N}\left(\prod_{i=1}^N s_i^{\alpha_i}\int_0^1 (1-\theta)^{m-1}\left[f_\alpha\left(x+\theta\left(sj\right)\right)-f_\alpha(x)\right]\mathrm{d}\theta\right)\mathrm{d}\mu_{\xi_n}(s)\right). \tag{1.19}$$

We further make

Remark 1.5. We further notice that

$$\theta_{r,n}^{[m]}(f;x)-f(x)=\sum_{j=0}^r \alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}\left(f\left(x+sj\right)-f(x)\right)\mathrm{d}\mu_{\xi_n}(s)=\sum_{j=0}^r \alpha_{j,r}^{[m]}\cdot$$

$$\int_{\mathbb{R}^N}\left[\sum_{\tilde{j}=1}^m \frac{g_{x+sj}^{(\tilde{j})}(0)}{\tilde{j}!}+\frac{1}{(m-1)!}\int_0^1 (1-\theta)^{m-1}\left(g_{x+sj}^{(m)}(\theta)-g_{x+sj}^{(m)}(0)\right)\mathrm{d}\theta\right]\mathrm{d}\mu_{\xi_n}(s). \tag{1.20}$$

That is

$$\begin{aligned}\Delta_{r,n}^{[m]}(x) &:= \theta_{r,n}^{[m]}(f;x)-f(x)-\sum_{j=0}^r \alpha_{j,r}^{[m]}\left(\int_{\mathbb{R}^N}\left(\sum_{\tilde{j}=1}^m \frac{g_{x+sj}^{(\tilde{j})}(0)}{\tilde{j}!}\right)\mathrm{d}\mu_{\xi_n}(s)\right)\\ &= \frac{1}{(m-1)!}\left(\sum_{j=0}^r \alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}\left(\int_0^1 (1-\theta)^{m-1}\right.\right.\\ &\qquad \left.\left.\times\left(g_{x+sj}^{(m)}(\theta)-g_{x+sj}^{(m)}(0)\right)\mathrm{d}\theta\right)\mathrm{d}\mu_{\xi_n}(s)\right)\\ &=: R_{r,n}^{[m]}.\end{aligned} \tag{1.21}$$

We see that

$$\Delta_{r,n}^{[m]}(x) = \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{j=1}^{r} \alpha_{j,r}^{[m]} \sum_{\tilde{j}=1}^{m} j^{\tilde{j}} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \tilde{j}}} \frac{c_{\alpha,n} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right)$$

$$= \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\tilde{j}=1}^{m} \left(\sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^{\tilde{j}} \right) \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \tilde{j}}} \frac{c_{\alpha,n} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right)$$

$$= \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \tilde{j}}} \frac{c_{\alpha,n} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right). \tag{1.22}$$

So we have proved that

$$\Delta_{r,n}^{[m]}(x) = \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \tilde{j}}} \frac{c_{\alpha,n} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right). \tag{1.23}$$

We also make

Remark 1.6. We observe that

$$R_{r,n}^{[m]} = m \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^m \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = m}\right)} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right).$$

$$\left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left(f_\alpha(x + \theta(sj)) - f_\alpha(x) \right) d\theta \right) d\mu_{\xi_n}(s) \right) \tag{1.24}$$

$$= m \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha|=m}\right)} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot$$

$$\left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^m \right.\right.$$

$$\left.\left. \times \left(f_\alpha \left(x + \theta \left(sj \right) \right) - f_\alpha (x) \right) \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right)$$

$$= m \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha|=m}\right)} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \cdot \right.\right.$$

$$\left.\left. \left[\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f_\alpha \left(x + \theta \left(sj \right) \right) + (-1)^r \binom{r}{0} f_\alpha (x) \right] \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right) \tag{1.25}$$

$$= m \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \cdot \right.\right.$$

$$\left.\left. \left[\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f_\alpha \left(x+\theta \left(sj \right) \right) \right] \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right). \tag{1.26}$$

We have proved that

$$R_{r,n}^{[m]} = m \sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot$$

$$\left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left(\Delta_{\theta s}^{r} f_\alpha (x) \right) \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right). \tag{1.27}$$

We further make

Remark 1.7. We observe that

$$\left|R_{r,n}^{[m]}\right| \overset{(1.27)}{\leq} m \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0 \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left|\Delta_{\theta s}^{r} f_\alpha(x)\right| \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right) \tag{1.28}$$

$$\leq m \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0 \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left\|\Delta_{\theta s}^{r} f_\alpha\right\|_\infty \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right)$$

$$\leq m \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0 \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \omega_r\left(f_\alpha, \theta \|s\|_2\right) \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right)$$

$$\leq \sum_{\substack{\alpha_1,\dots,\alpha_N \geq 0 \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \omega_r\left(f_\alpha, \|s\|_2\right) \right) \mathrm{d}\mu_{\xi_n}(s) \right). \tag{1.29}$$

So far we have established

$$\left|R_{r,n}^{[m]}\right| \le \sum_{\substack{\alpha_1,\dots,\alpha_N \ge 0 \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\, \omega_r\left(f_\alpha, \|s\|_2\right)\right) d\mu_{\xi_n}(s)\right)$$

$$= \sum_{\substack{\alpha_1,\dots,\alpha_N \ge 0 \\ |\alpha|=m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!}\right) \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\, \omega_r\left(f_\alpha, \xi_n \frac{\|s\|_2}{\xi_n}\right)\right) d\mu_{\xi_n}(s)\right) \tag{1.30}$$

(using the fact

$$\omega_r\left(f; \lambda u\right) \le (1+\lambda)^r\, \omega_r\left(f; u\right), \tag{1.31}$$

$\lambda, u > 0$, we get)

$$\le \sum_{\left(\substack{\alpha_1,\dots,\alpha_N \ge 0 \\ |\alpha|=m}\right)} \left(\frac{\omega_r\left(f_\alpha, \xi_n\right)}{\prod_{i=1}^{N} \alpha_i!}\right) \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)\right). \tag{1.32}$$

We have proved that

$$\left|R_{r,n}^{[m]}\right| \le \sum_{\left(\substack{\alpha_1,\dots,\alpha_N \ge 0 \\ |\alpha|=m}\right)} \left(\frac{\omega_r\left(f_\alpha, \xi_n\right)}{\prod_{i=1}^{N} \alpha_i!}\right) \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)\right). \tag{1.33}$$

We also make

Remark 1.8. Notice that for $|\alpha| = m$:

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \tag{1.34}$$

$$\le \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) < \infty,$$

by the assumption in Theorem 1.9, next.

Hence

$$|c_{\alpha,n}| \leq \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \mathrm{d}\mu_{\xi_n}(s) < \infty, \tag{1.35}$$

for all $\alpha : (\alpha_1, \ldots, \alpha_N) : |\alpha| := \sum \alpha_i = m, \alpha_i \in \mathbb{Z}^+$.

Hence also

$$\int_{\mathbb{R}^N} |s_i|^m \, \mathrm{d}\mu_{\xi_n}(s) < \infty, \tag{1.36}$$

for all $i = 1, \ldots, N$.

Let $1 \leq \tilde{j} \leq m-1$, then

$$\int_{\mathbb{R}^N} |s_i|^{\tilde{j}} \, \mathrm{d}\mu_{\xi_n}(s) \leq \left(\int_{\mathbb{R}^N} \left(|s_i|^{\tilde{j}} \right)^{\frac{m}{\tilde{j}}} \mathrm{d}\mu_{\xi_n}(s) \right)^{\frac{\tilde{j}}{m}}$$

$$= \left(\int_{\mathbb{R}^N} |s_i|^m \, \mathrm{d}\mu_{\xi_n}(s) \right)^{\frac{\tilde{j}}{m}} < \infty. \tag{1.37}$$

That is for $1 \leq \tilde{j} \leq m-1$, we obtain that

$$\int_{\mathbb{R}^N} |s_i|^{\tilde{j}} \, \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{1.38}$$

Let $\tilde{j} = 1, \ldots, m-1$; $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$. That is $\sum_{i=1}^N \left(\frac{\alpha_i}{\tilde{j}} \right) = 1$. Hence

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i} \, \mathrm{d}\mu_{\xi_n}(s) \leq \prod_{i=1}^{N} \left(\int_{\mathbb{R}^N} \left(|s_i|^{\alpha_i} \right)^{\frac{\tilde{j}}{\alpha_i}} \mathrm{d}\mu_{\xi_n}(s) \right)^{\frac{\alpha_i}{\tilde{j}}}$$

$$= \prod_{i=1}^{N} \left(\int_{\mathbb{R}^N} |s_i|^{\tilde{j}} \, \mathrm{d}\mu_{\xi_n}(s) \right)^{\frac{\alpha_i}{\tilde{j}}} < \infty. \tag{1.39}$$

Therefore, it holds

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i} \, \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{1.40}$$

Based on the above, we present

Theorem 1.9. *Let $m \in \mathbb{N}$, $f \in C^m(\mathbb{R}^N)$, $N \geq 1$, $x \in \mathbb{R}^N$. Assume $\left\| \frac{\partial^m f(\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \right\|_\infty < \infty$, for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N : |\alpha| := \sum_{j=1}^N \alpha_j = m$.*

Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ bounded sequence.

Suppose that for all $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = m$ *we have that*

$$u_{\xi_n} := \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{1.41}$$

For $\tilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^N \alpha_i = \tilde{j}$, *call*

$$c_{\alpha,n} := c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^N s_i^{\alpha_i} \mathrm{d}\mu_{\xi_n}(s_1, \ldots, s_N). \tag{1.42}$$

Then

i)

$$E_{r,n}^{[m]}(x) := \left| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \tilde{j}}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\prod_{i=1}^N \alpha_i!} \right) \right|$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha| = m} \right)} \frac{(\omega_r(f_\alpha, \xi_n))}{\left(\prod_{i=1}^N \alpha_i! \right)} \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \mathrm{d}\mu_{\xi_n}(s) \right), \tag{1.43}$$

$\forall\, x \in \mathbb{R}^N$.

ii)

$$\left\| E_{r,n}^{[m]} \right\|_\infty \leq R.H.S.\ (1.43). \tag{1.44}$$

Given that $\xi_n \to 0$, *as* $n \to \infty$, *and* u_{ξ_n} *is uniformly bounded, then we derive that* $\left\| E_{r,n}^{[m]} \right\|_\infty \to 0$ *with rates.*

iii) It holds also that

$$\left\| \theta_{r,n}^{[m]}(f) - f \right\|_\infty \leq \sum_{\tilde{j}=1}^m \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha| = \tilde{j}}} \frac{\left| c_{\alpha,n,\tilde{j}} \right| \| f_\alpha \|_\infty}{\prod_{i=1}^N \alpha_i!} \right) + R.H.S.\ (1.43), \tag{1.45}$$

given that $\|f_\alpha\|_\infty < \infty$, *for all* $\alpha : |\alpha| = \tilde{j}$, $\tilde{j} = 1, \ldots, m$. *Furthermore, as* $\xi_n \to 0$ *when* $n \to \infty$, *assuming that* $c_{\alpha,n,\tilde{j}} \to 0$, *while* u_{ξ_n} *is uniformly bounded, we conclude that*

$$\left\| \theta_{r,n}^{[m]}(f) - f \right\|_\infty \to 0 \tag{1.46}$$

with rates.

Case $m = 0$.

Remark 1.10. Here $f \in C_B\left(\mathbb{R}^N\right)$ (bounded and continuous functions). We notice that

$$\theta_{r,n}^{[0]}(f;x) - f(x)$$

$$= \sum_{j=0}^{r} \alpha_{j,r}^{[0]} \int_{\mathbb{R}^N} (f(x+js) - f(x))\, d\mu_{\xi_n}(s)$$

$$= \int_{\mathbb{R}^N} \left(\sum_{j=0}^{r} \alpha_{j,r}^{[0]} f(x+js) - \left(\sum_{j=0}^{r} \alpha_{j,r}^{[0]} \right) f(x) \right) d\mu_{\xi_n}(s) \tag{1.47}$$

$$= \int_{\mathbb{R}^N} \left(\sum_{j=1}^{r} \alpha_{j,r}^{[0]} f(x+js) - \left(\sum_{j=1}^{r} \alpha_{j,r}^{[0]} \right) f(x) \right) d\mu_{\xi_n}(s) \tag{1.48}$$

$$= \int_{\mathbb{R}^N} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f(x+js) - \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} \right) f(x) \right) d\mu_{\xi_n}(s)$$

$$= \int_{\mathbb{R}^N} \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f(x+js) + (-1)^r \binom{r}{0} f(x) \right) d\mu_{\xi_n}(s)$$

$$= \int_{\mathbb{R}^N} \left(\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x+js) \right) d\mu_{\xi_n}(s) = \int_{\mathbb{R}^N} \left(\Delta_s^r f(x) \right) d\mu_{\xi_n}(s). \tag{1.49}$$

We established that

$$\theta_{r,n}^{[0]}(f;x) - f(x) = \int_{\mathbb{R}^N} \left(\Delta_s^r f(x) \right) d\mu_{\xi_n}(s). \tag{1.50}$$

Consequently, it holds

$$\left|\theta_{r,n}^{[0]}(f;x)-f(x)\right| \leq \int_{\mathbb{R}^N} |\Delta_s^r f(x)|\, d\mu_{\xi_n}(s)$$
$$\leq \int_{\mathbb{R}^N} \|\Delta_s^r f\|_\infty \, d\mu_{\xi_n}(s) \leq \int_{\mathbb{R}^N} \omega_r\left(f, \|s\|_2\right) d\mu_{\xi_n}(s)$$
$$= \int_{\mathbb{R}^N} \omega_r\left(f, \xi_n \frac{\|s\|_2}{\xi_n}\right) d\mu_{\xi_n}(s)$$
$$\leq \omega_r(f,\xi_n) \int_{\mathbb{R}^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s). \tag{1.51}$$

So we proved

$$\left|\theta_{r,n}^{[0]}(f;x)-f(x)\right| \leq \left(\int_{\mathbb{R}^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)\right) \omega_r(f,\xi_n), \tag{1.52}$$

$\forall\, x \in \mathbb{R}$.

Based on the above, we present

Theorem 1.11. *Let* $f \in C_B\left(\mathbb{R}^N\right)$*,* $N \geq 1$*. Then*

$$\left\|\theta_{r,n}^{[0]} f - f\right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s)\right) \omega_r(f,\xi_n), \tag{1.53}$$

under the assumption

$$\Phi_{\xi_n} := \int_{\mathbb{R}^N} \left(1+\frac{\|s\|_2}{\xi_n}\right)^r d\mu_{\xi_n}(s) < \infty. \tag{1.54}$$

As $n \to \infty$ *and* $\xi_n \to 0$*, given that* Φ_{ξ_n} *are uniformly bounded, we derive*

$$\left\|\theta_{r,n}^{[0]} f - f\right\|_\infty \to 0 \tag{1.55}$$

with rates.

1.3 Applications

Let all entities as in Sect. 1.2. We define the following specific operators:

(a) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}(f;x_1,\ldots,x_N) := \frac{1}{(2\xi_n)^N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{1.56}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j, x_2+s_2 j, \ldots, x_N+s_N j)\,\mathrm{e}^{-\frac{\left(\sum_{i=1}^{N}|s_i|\right)}{\xi_n}}\,\mathrm{d}s_1\ldots\mathrm{d}s_N.$$

Observe that

$$\frac{1}{(2\xi_n)^N}\int_{\mathbb{R}^N}\mathrm{e}^{-\frac{\left(\sum_{i=1}^{N}|s_i|\right)}{\xi_n}}\,\mathrm{d}s_1\ldots\mathrm{d}s_N = 1, \tag{1.57}$$

see [1].

(b) The general multivariate Gauss–Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f;x_1,\ldots,x_N) := \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{1.58}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j, x_2+s_2 j, \ldots, x_N+s_N j)\,\mathrm{e}^{-\frac{\left(\sum_{i=1}^{N}s_i^2\right)}{\xi_n}}\,\mathrm{d}s_1\ldots\mathrm{d}s_N.$$

Notice that

$$\frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\int_{\mathbb{R}^N}\mathrm{e}^{-\frac{\left(\sum_{i=1}^{N}s_i^2\right)}{\xi_n}}\,\mathrm{d}s_1\ldots\mathrm{d}s_N = 1, \tag{1.59}$$

see [7].

(c) The general multivariate Poisson–Cauchy singular integral operators:

$$U_{r,n}^{[m]}(f;x_1,\ldots,x_N) := W_n^N\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{1.60}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j,\ldots,x_N+s_N j)\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}\,\mathrm{d}s_1\ldots\mathrm{d}s_N,$$

with $\alpha\in\mathbb{N}$, $\beta>\frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta)\,\alpha\xi_n^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}, \tag{1.61}$$

see [8].

Observe that

$$W_n^N \int_{\mathbb{R}^N} \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^{\beta}} ds_1 \ldots ds_N = 1, \tag{1.62}$$

see [8, 13], p. 397, formula 595.

(d) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{1.63}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1 j, \ldots, x_N+s_N j) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 \ldots ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta} \pi (-1)^{\beta} \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}, \tag{1.64}$$

see [9, 10], p. 210, item 1033.

Notice that

$$\lambda_n^{-N} \int_{\mathbb{R}^N} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 \ldots ds_N = 1, \tag{1.65}$$

see also [9, 10], p. 210, item 1033.

One can apply Theorems 1.9 and 1.11 to operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results.

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Chapter 2
L_p-Approximation by General Multivariate Singular Integral Operators

In this chapter, we present the L_p, $1 \leq p < \infty$ approximation properties of general multivariate singular integral operators over R^N, $N \geq 1$. We give their convergence to the unit operator with rates. The established inequalities involve the multivariate higher order modulus of smoothness. We list the multivariate Picard, Gauss–Weierstrass, Poisson Cauchy and trigonometric singular integral operators where this theory can be applied directly. This chapter is based on [3].

2.1 Introduction

The rate of L_p, $1 \leq p < \infty$ convergence of univariate singular integral operators has been studied in [1, 7–9], see also the related [11, 12], and these works motivate the current chapter. Here, we consider some very general multivariate singular integral operators over $\mathbb{R}^N$, $N \geq 1$, and we study the degree of approximation to the unit operator with rates over smooth functions. The derived related inequalities are involving the multivariate higher modulus of smoothness with respect to $\|\cdot\|_p$. See Theorems 2.4, 2.6, 2.8, 2.10. We mention particular operators that fulfill this theory. The discussed linear operators are not in general positive, see [2]. Other motivation comes from [4, 5].

2.2 Main Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \ldots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-m}, & \text{if } j = 0, \end{cases} \tag{2.1}$$

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and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \ldots, m \in \mathbb{N}. \tag{2.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{2.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{2.4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]} (f; x_1, \ldots, x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) d\mu_{\xi_n} (s), \tag{2.5}$$

where $s := (s_1, \ldots, s_N)$, $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

The operators $\theta_{r,n}^{[m]}$ preserve constants, see [2].

Here, we deal with $f \in C^m (\mathbb{R}^N)$, $m \in \mathbb{Z}^+$, with $f_\alpha \in L_p (\mathbb{R}^N)$, $|\alpha| = m \in \mathbb{Z}^+$, $p \geq 1$; where f_α denotes the mixed partial $\frac{\partial^{\tilde{j}} f(\cdot,\ldots,\cdot)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$, $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N$: $|\alpha| := \sum_{j=1}^{N} \alpha_j = \tilde{j}$, $\tilde{j} = 1, \ldots, m$.

We need

Definition 2.1 (see also [6]).

We call

$$\Delta_u^r f (x) := \Delta_{u_1, u_2, \ldots, u_N}^r f (x_1, \ldots, x_N) \tag{2.6}$$
$$:= \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f (x_1 + j u_1, x_2 + j u_2, \ldots, x_N + j u_N).$$

Let $p \geq 1$, the modulus of smoothness of order r is given by

$$\omega_r (f; h)_p := \sup_{\|u\|_2 \leq h} \left\| \Delta_u^r (f) \right\|_p, \tag{2.7}$$

$h > 0$.

I) First, we consider the case of $m \in \mathbb{N}$, $p > 1$.

We make

Remark 2.2. For $\tilde{j} = 1, \ldots, m$, and $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \tilde{j}$, under the assumption of Theorem 2.4,

$$c_{\alpha,n,\tilde{j}} := c_{\alpha,n} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, d\mu_{\xi_n}(s_1, \ldots, s_N) \in \mathbb{R}, \tag{2.8}$$

see also [2].

From [2], we obtain

$$E_{r,n}^{[m]}(x) := \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \tag{2.9}$$

$$= m \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \right.\right.$$

$$\left.\left. \times \left(\Delta_{\theta s}^r f_\alpha(x) \right) d\theta \right) d\mu_{\xi_n}(s) \right)$$

$$=: R_{r,n}^{[m]}(x), \quad \forall\, x \in \mathbb{R}^N. \tag{2.10}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| E_{r,n}^{[m]}(x) \right|^p = \left| R_{r,n}^{[m]}(x) \right|^p$$

$$\left(set\ c_1 := \left(m \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \right)^p \right)$$

$$= c_1 \left| \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left(\Delta_{\theta s}^r f_\alpha(x) \right) d\theta \right) d\mu_{\xi_n}(s) \right|^p$$

$$\leq c_1 \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^r f_\alpha(x) \right| d\theta \right) d\mu_{\xi_n}(s) \right)^p. \tag{2.11}$$

Hence, we have

$$I_1 := \int_{\mathbb{R}^N} \left| E_{r,n}^{[m]}(x) \right|^p \mathrm{d}x \tag{2.12}$$

$$\leq c_1 \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} |\Delta_{\theta s}^r f_\alpha(x)| \,\mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right)^p \mathrm{d}x$$

$$\left(\mathit{Call}\ 0 \leq \gamma(s,x) := \prod_{i=1}^N |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} |\Delta_{\theta s}^r f_\alpha(x)| \,\mathrm{d}\theta \right)$$

$$= c_1 \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \gamma(s,x) \,\mathrm{d}\mu_{\xi_n}(s) \right)^p \mathrm{d}x =: I_2. \tag{2.13}$$

Therefore, it holds

$$I_2 \leq c_1 \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \gamma^p(s,x) \,\mathrm{d}\mu_{\xi_n}(s) \right) \mathrm{d}x =: I_3. \tag{2.14}$$

But we have

$$\gamma(s,x) \leq \prod_{i=1}^N |s_i|^{\alpha_i} \left(\int_0^1 \left((1-\theta)^{m-1} \right)^q \mathrm{d}\theta \right)^{\frac{1}{q}} \left(\int_0^1 |\Delta_{\theta s}^r f_\alpha(x)|^p \,\mathrm{d}\theta \right)^{\frac{1}{p}} \tag{2.15}$$

$$= \prod_{i=1}^N |s_i|^{\alpha_i} \frac{1}{(q(m-1)+1)^{\frac{1}{q}}} \left(\int_0^1 |\Delta_{\theta s}^r f_\alpha(x)|^p \,\mathrm{d}\theta \right)^{\frac{1}{p}}.$$

Hence, we obtain

$$\gamma^p(s,x) \leq \prod_{i=1}^N |s_i|^{\alpha_i p} \frac{1}{(q(m-1)+1)^{\frac{p}{q}}} \left(\int_0^1 |\Delta_{\theta s}^r f_\alpha(x)|^p \,\mathrm{d}\theta \right). \tag{2.16}$$

Thus, we get

$$I_3 \leq c_2 \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \prod_{i=1}^N |s_i|^{\alpha_i p} \left(\int_0^1 |\Delta_{\theta s}^r f_\alpha(x)|^p \,\mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) \right) \mathrm{d}x \tag{2.17}$$

$$\left(set\ c_2 := c_1 \cdot \frac{1}{(q\,(m-1)+1)^{\frac{p}{q}}} \right)$$

$$= c_2 \int_{\mathbb{R}^N} \left(\int_0^1 \left(\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left|\Delta_{\theta s}^r f_\alpha(x)\right|^p \mathrm{d}x \right) \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s)$$

$$= c_2 \int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left(\int_0^1 \left(\int_{\mathbb{R}^N} \left|\Delta_{\theta s}^r f_\alpha(x)\right|^p \mathrm{d}x \right) \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s) =: I_4. \quad (2.18)$$

Consequently, we derive

$$I_4 \le c_2 \int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i p} \left(\int_0^1 \left(\omega_r\left(f_\alpha; \theta \|s\|_2\right)_p \right)^p \mathrm{d}\theta \right) \mathrm{d}\mu_{\xi_n}(s)$$

$$\le c_2 \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \omega_r\left(f_\alpha; \|s\|_2\right)_p^p \mathrm{d}\mu_{\xi_n}(s)$$

$$= c_2 \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \omega_r\left(f_\alpha; \xi_n \frac{\|s\|_2}{\xi_n}\right)_p^p \mathrm{d}\mu_{\xi_n}(s) \quad (2.19)$$

$$\left(by\ \omega_r(f, \lambda h)_p \le (1+\lambda)^r\, \omega_r(f, h)_p,\ for\ any\ h, \lambda > 0,\ p \ge 1 \right)$$

$$\le c_2 \omega_r\left(f_\alpha; \xi_n\right)_p^p \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i p} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{\mathrm{rp}} \mathrm{d}\mu_{\xi_n}(s) \right). \quad (2.20)$$

We have proved that

$$\int_{\mathbb{R}^N} \left| E_{r,n}^{[m]}(x) \right|^p \mathrm{d}x \le \left[\left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \right) \cdot \left(\frac{m}{(q\,(m-1)+1)^{\frac{1}{q}}} \right) \right]^p \cdot$$

$$\omega_r\left(f_\alpha; \xi_n\right)_p^p \left(\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p \mathrm{d}\mu_{\xi_n}(s) \right). \quad (2.21)$$

Thus, we derive ($p > 1$)

$$\left\| E_{r,n}^{[m]}(x) \right\|_p \leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \right) \cdot$$

$$\left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \omega_r(f_\alpha; \xi_n)_p . \tag{2.22}$$

We make

Remark 2.3. Notice that ($p > 1$)

$$\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \tag{2.23}$$

$$\leq \left[\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} < \infty,$$

by assumption of Theorem 2.4.

As in [2] then we get that

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |s_i|^{\alpha_i} d\mu_{\xi_n}(s) < \infty. \tag{2.24}$$

Hence, $c_{\alpha,n,\tilde{j}} \in \mathbb{R}$.

From the above, we have proved

Theorem 2.4. *Let* $f \in C^m\left(\mathbb{R}^N\right)$, $m \in \mathbb{N}$, $N \geq 1$, *with* $f_\alpha \in L_p\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *for* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence. Assume for all* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$ *that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{2.25}$$

For $\tilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \tilde{j}$, *call*

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, d\mu_{\xi_n}(s). \tag{2.26}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_p = \left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \right\|_{p,x} \tag{2.27}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \cdot$$

$$\left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \omega_r(f_\alpha, \xi_n)_p.$$

As $n \to \infty$ *and* $\xi_n \to 0$, *by (2.27), we obtain that* $\left\| E_{r,n}^{[m]} \right\|_p \to 0$ *with rates.*

One also finds by (2.27) that

$$\left\| \theta_{r,n}^{[m]}(f;x) - f(x) \right\|_{p,x} \leq$$

$$\sum_{\tilde{j}=1}^{m} \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{\left| c_{\alpha,n,\tilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i!} \| f_\alpha \|_p \right) + R.H.S. \ (2.27),$$

given that $\| f_\alpha \|_p < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \ldots, m$.

Assuming that $c_{\alpha,n,\tilde{j}} \to 0$, $\xi_n \to 0$, *as* $n \to \infty$, *we get* $\left\| \theta_{r,n}^{[m]}(f) - f \right\|_p \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *the unit operator, in* L_p *norm, with rates.*

II) Case of $m = 0$, $p > 1$.
We make

Remark 2.5. In [2] we proved that

$$\theta_{r,n}^{[0]}(f;x) - f(x) = \int_{\mathbb{R}^N} \left(\Delta_s^r f(x) \right) d\mu_{\xi_n}(s). \tag{2.28}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Hence

$$\left|\theta_{r,n}^{[0]}(f;x) - f(x)\right|^p \leq \left(\int_{\mathbb{R}^N} |\Delta_s^r f(x)| \, d\mu_{\xi_n}(s)\right)^p \tag{2.29}$$
$$\leq \int_{\mathbb{R}^N} |\Delta_s^r f(x)|^p \, d\mu_{\xi_n}(s).$$

And it holds

$$\int_{\mathbb{R}^N} \left|\theta_{r,n}^{[0]}(f;x) - f(x)\right|^p dx \tag{2.30}$$
$$\leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |\Delta_s^r f(x)|^p \, d\mu_{\xi_n}(s)\right) dx$$
$$= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |\Delta_s^r f(x)|^p \, dx\right) d\mu_{\xi_n}(s) \tag{2.31}$$
$$\leq \int_{\mathbb{R}^N} \omega_r(f, \|s\|_2)_p^p \, d\mu_{\xi_n}(s)$$
$$= \int_{\mathbb{R}^N} \omega_r\left(f, \xi_n \frac{\|s\|_2}{\xi_n}\right)_p^p d\mu_{\xi_n}(s) \tag{2.32}$$
$$\leq \omega_r(f, \xi_n)_p^p \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s).$$

Therefore, we obtain

$$\left\|\theta_{r,n}^{[0]}(f) - f\right\|_p \tag{2.33}$$
$$\leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s)\right)^{\frac{1}{p}} \omega_r(f, \xi_n)_p.$$

We proved

Theorem 2.6. *Let* $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$*;* $N \geq 1$*;* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$*,* $(\xi_n)_{n \in \mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^{rp} d\mu_{\xi_n}(s) < \infty. \tag{2.34}$$

Then

$$\left\| \theta_{r,n}^{[0]}(f) - f \right\|_p \tag{2.35}$$
$$\leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{\mathrm{rp}} d\mu_{\xi_n}(s) \right)^{\frac{1}{p}} \omega_r(f, \xi_n)_p .$$

As $\xi_n \to 0$, *when* $n \to \infty$, *we derive* $\left\| \theta_{r,n}^{[0]}(f) - f \right\|_p \to 0$, *i.e.* $\theta_{r,n}^{[0]} \to I$, *the unit operator, in* L_p *norm.*

III) Next follows the case $m = 0$, $p = 1$.
We make

Remark 2.7. As before we have

$$\left| \theta_{r,n}^{[0]}(f; x) - f(x) \right| \leq \int_{\mathbb{R}^N} \left| \Delta_s^r f(x) \right| d\mu_{\xi_n}(s) . \tag{2.36}$$

Hence

$$\left\| \theta_{r,n}^{[0]}(f) - f \right\|_1 = \int_{\mathbb{R}^N} \left| \theta_{r,n}^{[0]}(f; x) - f(x) \right| dx \tag{2.37}$$

$$\leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \Delta_s^r f(x) \right| d\mu_{\xi_n}(s) \right) dx \tag{2.38}$$

$$= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \Delta_s^r f(x) \right| dx \right) d\mu_{\xi_n}(s)$$

$$\leq \int_{\mathbb{R}^N} \omega_r(f, \|s\|_2)_1 \, d\mu_{\xi_n}(s)$$

$$= \int_{\mathbb{R}^N} \omega_r \left(f, \xi_n \frac{\|s\|_2}{\xi_n} \right)_1 d\mu_{\xi_n}(s) \tag{2.39}$$

$$\leq \omega_r(f, \xi_n)_1 \int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) .$$

We have proved

Theorem 2.8. *Let* $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$, $N \geq 1$. *Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$, $(\xi_n)_{n \in \mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{2.40}$$

Then

$$\left\| \theta_{r,n}^{[0]}(f) - f \right\|_1 \tag{2.41}$$
$$\leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \omega_r(f, \xi_n)_1 .$$

As $\xi_n \to 0$*, we get* $\theta_{r,n}^{[0]} \to I$ *in* L_1 *norm.*

IV) Case of $m \in \mathbb{N}$, $p = 1$.
We make

Remark 2.9. We have

$$\left\| E_{r,n}^{[m]} \right\|_1 = \int_{\mathbb{R}^N} \left| E_{r,n}^{[m]}(x) \right| dx \leq m \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \cdot$$
$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \int_0^1 (1-\theta)^{m-1} \left| \Delta_{\theta s}^r f_\alpha(x) \right| d\theta \right) d\mu_{\xi_n}(s) \right) dx \tag{2.42}$$

$$\left(set\ \overline{c_1} := m \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^N \alpha_i!} \right) \right)$$
$$= \overline{c_1} \int_{\mathbb{R}^N} \prod_{i=1}^N |s_i|^{\alpha_i} \left(\int_0^1 (1-\theta)^{m-1} \left(\int_{\mathbb{R}^N} \left| \Delta_{\theta s}^r f_\alpha(x) \right| dx \right) d\theta \right) d\mu_{\xi_n}(s)$$
$$\leq \overline{c_1} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \left(\int_0^1 (1-\theta)^{m-1} \omega_r(f_\alpha, \theta \|s\|_2)_1 \, d\theta \right) d\mu_{\xi_n}(s) \tag{2.43}$$
$$\leq \frac{\overline{c_1}}{m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i} \right) \omega_r(f_\alpha, \|s\|_2)_1 \, d\mu_{\xi_n}(s)$$

$$= \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \omega_r \left(f_\alpha, \xi_n \frac{\|s\|_2}{\xi_n} \right)_1 \mathrm{d}\mu_{\xi_n}(s) \tag{2.44}$$

$$\leq \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r (f_\alpha, \xi_n)_1 \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \mathrm{d}\mu_{\xi_n}(s). \tag{2.45}$$

We have proved

Theorem 2.10. *Let $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$, with $f_\alpha \in L_1(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence. Suppose for all $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$ that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right) \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{2.46}$$

For $\tilde{j} = 1, \ldots, m$, and $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \tilde{j}$, call

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \mathrm{d}\mu_{\xi_n}(s). \tag{2.47}$$

Then

$$\left\| E_{r,n}^{[m]} \right\|_1 = \left\| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{1,x} \tag{2.48}$$

$$\leq \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \omega_r (f_\alpha, \xi_n)_1 \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \mathrm{d}\mu_{\xi_n}(s).$$

As $\xi_n \to 0$, we get $\left\| E_{r,n}^{[m]} \right\|_1 \to 0$ with rates.

From (2.48) we get

$$\left\| \theta_{r,n}^{[m]} f - f \right\|_1$$

$$\leq \sum_{\tilde{j}=1}^{m} \left| \delta_{\tilde{j},r}^{[m]} \right| \left(\sum_{|\alpha|=\tilde{j}} \frac{\left| c_{\alpha,n,\tilde{j}} \right|}{\prod_{i=1}^{N} \alpha_i !} \| f_\alpha \|_1 \right) + R.H.S.\ (2.48),$$

given that $\| f_\alpha \|_1 < \infty$, $|\alpha| = \tilde{j}$, $\tilde{j} = 1, \ldots, m$.

As $n \to \infty$, *assuming* $\xi_n \to 0$ *and* $c_{\alpha,n,\tilde{j}} \to 0$, *we obtain* $\left\| \theta_{r,n}^{[m]} f - f \right\|_1 \to 0$, *that is* $\theta_{r,n}^{[m]} \to I$ *in* L_1 *norm, with rates.*

2.3 Applications

Let all entities as in Sect. 2.2. We define the following specific operators:

(a) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]} (f; x_1, \ldots, x_N) := \frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{2.49}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N} |s_i| \right)}{\xi_n}} ds_1 \ldots ds_N.$$

Notice that

$$\frac{1}{(2\xi_n)^N} \int_{\mathbb{R}^N} e^{-\frac{\left(\sum_{i=1}^{N} |s_i| \right)}{\xi_n}} ds_1 \ldots ds_N = 1, \tag{2.50}$$

see [1].

(b) The general multivariate Gauss–Weierstrass singular integral operators:

$$W_{r,n}^{[m]} (f; x_1, \ldots, x_N) := \frac{1}{\left(\sqrt{\pi \xi_n} \right)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{2.51}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N} s_i^2 \right)}{\xi_n}} ds_1 \ldots ds_N.$$

Observe that

$$\frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \int_{\mathbb{R}^N} e^{-\frac{\left(\sum_{i=1}^N s_i^2\right)}{\xi_n}} ds_1 \dots ds_N = 1, \tag{2.52}$$

see [7].

(c) The general multivariate Poisson–Cauchy singular integral operators:

$$U_{r,n}^{[m]}(f; x_1, \dots, x_N) := W_n^N \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{2.53}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, \dots, x_N + s_N j) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 \dots ds_N,$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta)\, \alpha \xi_n^{2\alpha\beta - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}, \tag{2.54}$$

see [8].

Notice that

$$W_n^N \int_{\mathbb{R}^N} \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 \dots ds_N = 1, \tag{2.55}$$

see [8, 13], p. 397, formula 595.

(d) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}(f; x_1, \dots, x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{2.56}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, \dots, x_N + s_N j) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 \dots ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta} \pi (-1)^\beta \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta - 1}}{(\beta - k)!\,(\beta + k)!}, \tag{2.57}$$

see [9, 10], p. 210, item 1033.

Observe that

$$\lambda_n^{-N} \int_{\mathbb{R}^N} \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 \ldots ds_N = 1, \tag{2.58}$$

see also [9, 10], p. 210, item 1033.

One can apply Theorems 2.4, 2.6, 2.8, and 2.10 to operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results.

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Chapter 3
Global Smoothness Preservation and Simultaneous Approximation by Multivariate General Singular Integrals

In this chapter, we continue with the study of multivariate smooth general singular integral operators over R^N, $N \geq 1$, regarding their simultaneous global smoothness preservation property with respect to the L_p norm, $1 \leq p \leq \infty$, by involving multivariate higher order moduli of smoothness. Also, we present their multivariate simultaneous approximation to the unit operator with rates. The derived multivariate Jackson-type inequalities are almost sharp containing elegant constants, and they reflect the high order of differentiability of the engaged function. In the uniform case of global smoothness we prove optimality. At the end we list the multivariate Picard, Gauss–Weierstrass, Poisson–Cauchy and Trigonometric singular integral operators as applicators of this general theory. This chapter relies on [4].

3.1 Introduction

The main motivation for this chapter comes from [1, 5, 6]. We give here the multivariate simultaneous global smoothness preservation property of multivariate general smooth singular integral operators. We study also the simultaneous L_p, $1 \leq p \leq \infty$, approximation of these operators to the unit operator with rates. See Theorems 3.2, 3.4, Proposition 3.5, Theorem 3.8, Corollary 3.9 and Theorems 3.10–3.15. At the end, we list specific operators that fulfill our theory. One can find many interesting convergence properties based on these results.

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3.2 Main Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \dots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-m}, & \text{if } j = 0, \end{cases} \tag{3.1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \dots, m \in \mathbb{N}. \tag{3.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{3.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{3.4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]} (f; x_1, \dots, x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, \dots, x_N + s_N j) \, d\mu_{\xi_n} (s), \tag{3.5}$$

where $s := (s_1, \dots, s_N)$, $x := (x_1, \dots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

Above operators $\theta_{r,n}^{[m]}$ are not in general positive operators and they preserve constants, see [2].

Definition 3.1. Let $f \in C\left(\mathbb{R}^N\right)$, $N \geq 1$, $m \in \mathbb{N}$, the mth modulus of smoothness for $1 \leq p \leq \infty$, is given by

$$\omega_m (f; h)_p := \sup_{\|t\|_2 \leq h} \left\| \Delta_t^m (f) \right\|_{p,x}, \tag{3.6}$$

$h > 0$, where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jt). \tag{3.7}$$

Denote

$$\omega_m (f; h)_\infty = \omega_m (f, h). \tag{3.8}$$

Above, $x, t \in \mathbb{R}^N$.

We present the general global smoothness preservation result

Theorem 3.2. *We assume $\theta_{r,n}^{[\tilde{m}]}(f; x) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}$. Let $h > 0$, $f \in C(\mathbb{R}^N)$, $N \geq 1$.*

i) Assume $\omega_m(f, h) < \infty$. Then

$$\omega_m \left(\theta_{r,n}^{[\tilde{m}]} f, h\right) \leq \left(\sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right|\right) \omega_m (f, h). \tag{3.9}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$. Then

$$\omega_m \left(\theta_{r,n}^{[\tilde{m}]} f, h\right)_1 \leq \left(\sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right|\right) \omega_m (f, h)_1. \tag{3.10}$$

iii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p > 1$. Then

$$\omega_m \left(\theta_{r,n}^{[\tilde{m}]} f, h\right)_p \leq \left(\sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right|\right) \omega_m (f, h)_p. \tag{3.11}$$

Proof. We recall ($x \in \mathbb{R}^N$, $N \geq 1$)

$$\theta_{r,n}^{[\tilde{m}]}(f; x) = \sum_{\tilde{j}=0}^{r} \alpha_{\tilde{j},r}^{[\tilde{m}]} \int_{\mathbb{R}^N} f\left(x + s\tilde{j}\right) \mathrm{d}\mu_{\xi_n}(s).$$

We see ($t \in \mathbb{R}^N$)

$$\Delta_t^m \left(\theta_{r,n}^{[\tilde{m}]}(f; x)\right) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \theta_{r,n}^{[\tilde{m}]}(f; x + jt) \tag{3.12}$$

$$= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \int_{\mathbb{R}^N} \left(\sum_{\tilde{j}=0}^{r} \alpha_{\tilde{j},r}^{[\tilde{m}]} f\left(x + jt + \tilde{j}k\right)\right) \mathrm{d}\mu_{\xi_n}(k) \tag{3.13}$$

$$= \sum_{\tilde{j}=0}^{r} \alpha_{\tilde{j},r}^{[\tilde{m}]} \int_{\mathbb{R}^N} \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f\left(x+jt+\tilde{j}k\right) \right) d\mu_{\xi_n}(k)$$

$$= \sum_{\tilde{j}=0}^{r} \alpha_{\tilde{j},r}^{[\tilde{m}]} \int_{\mathbb{R}^N} \left(\Delta_t^m f\left(x+\tilde{j}k\right)\right) d\mu_{\xi_n}(k). \tag{3.14}$$

That is we find

$$\Delta_t^m \left(\theta_{r,n}^{[\tilde{m}]}(f;x)\right) = \sum_{\tilde{j}=0}^{r} \alpha_{\tilde{j},r}^{[\tilde{m}]} \int_{\mathbb{R}^N} \left(\Delta_t^m f\left(x+\tilde{j}k\right)\right) d\mu_{\xi_n}(k). \tag{3.15}$$

Therefore

$$\left|\Delta_t^m \left(\theta_{r,n}^{[\tilde{m}]}(f;x)\right)\right| \le \sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right| \int_{\mathbb{R}^N} \left| \Delta_t^m f\left(x+\tilde{j}k\right)\right| d\mu_{\xi_n}(k). \tag{3.16}$$

(a) From the last (3.16), we derive

$$\omega_m\left(\theta_{r,n}^{[\tilde{m}]} f, h\right) \le \sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right| \int_{\mathbb{R}^N} \omega_m(f,h)\, d\mu_{\xi_n}(k) \tag{3.17}$$

$$= \left(\sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right| \right) \omega_m(f,h),$$

proving the claim (3.9).

(b) We see that

$$\int_{\mathbb{R}^N} \left|\Delta_t^m \left(\theta_{r,n}^{[\tilde{m}]}(f;x)\right)\right| dx \le \sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right| \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \Delta_t^m f\left(x+\tilde{j}k\right)\right| dx \right)$$

$$\times\, d\mu_{\xi_n}(k)$$

$$= \sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right| \int_{\mathbb{R}^N} \|\Delta_t^m f\|_1\, d\mu_{\xi_n}(k)$$

$$= \left(\sum_{\tilde{j}=0}^{r} \left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right| \right) \|\Delta_t^m f\|_1. \tag{3.18}$$

So we derive

$$\left\| \Delta_t^m \left(\theta_{r,n}^{[\tilde{m}]} (f) \right) \right\|_1 \le \left(\sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \right) \| \Delta_t^m f \|_1 , \tag{3.19}$$

which implies

$$\omega_m \left(\theta_{r,n}^{[\tilde{m}]} f, h \right)_1 \le \left(\sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \right) \omega_m (f, h)_1 , \tag{3.20}$$

proving the claim (3.10).

(c) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left\| \Delta_t^m \left(\theta_{r,n}^{[\tilde{m}]} (f; x) \right) \right\|_{p,x} \le \sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \left\| \int_{\mathbb{R}^N} \left| \Delta_t^m f \left(x + \tilde{j}k \right) \right| d\mu_{\xi_n} (k) \right\|_{p,x} \tag{3.21}$$

$$= \sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \Delta_t^m f \left(x + \tilde{j}k \right) \right| \times d\mu_{\xi_n} (k) \right)^p dx \right)^{\frac{1}{p}} \tag{3.22}$$

$$\le \sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \Delta_t^m f \left(x + \tilde{j}k \right) \right|^p \times d\mu_{\xi_n} (k) \right) dx \right)^{\frac{1}{p}}$$

$$= \sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \left(\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \left| \Delta_t^m f \left(x + \tilde{j}k \right) \right|^p dx \right) \times d\mu_{\xi_n} (k) \right)^{\frac{1}{p}}$$

$$= \sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \left(\int_{\mathbb{R}^N} \| \Delta_t^m f \|_p^p \, d\mu_{\xi_n} (k) \right)^{\frac{1}{p}}$$

$$= \left(\sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \right) \| \Delta_t^m f \|_p . \tag{3.23}$$

That is

$$\left\| \Delta_t^m \left(\theta_{r,n}^{[\tilde{m}]} (f) \right) \right\|_p \leq \left(\sum_{\tilde{j}=0}^{r} \left| \alpha_{\tilde{j},r}^{[\tilde{m}]} \right| \right) \| \Delta_t^m f \|_p . \tag{3.24}$$

Now clearly (3.24) proves (3.11). □

Remark 3.3. Let $r = 1$, then $\alpha_{0,1}^{[m]} = 0$, $\alpha_{1,1}^{[m]} = 1$. Hence

$$\theta_{1,n}^{[\tilde{m}]} (f; x) = \int_{\mathbb{R}^N} f (x + s) \, d\mu_{\xi_n} (s) =: \theta_n (f; x) . \tag{3.25}$$

By Theorem 3.2, we get

Theorem 3.4. *We suppose $\theta_n (f; x) \in \mathbb{R}$, $\forall \; x \in \mathbb{R}$. Let $h > 0$, $f \in C(\mathbb{R}^N)$, $N \geq 1$.*

i) Assume $\omega_m (f, h) < \infty$. Then

$$\omega_m (\theta_n f, h) \leq \omega_m (f, h) . \tag{3.26}$$

ii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$. Then

$$\omega_m (\theta_n f, h)_1 \leq \omega_m (f, h)_1 . \tag{3.27}$$

iii) Assume $f \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $p > 1$. Then

$$\omega_m (\theta_n f, h)_p \leq \omega_m (f, h)_p . \tag{3.28}$$

Next, we get an optimality result

Proposition 3.5. *Above inequality (3.26):*

$$\omega_m (\theta_n f, h) \leq \omega_m (f, h)$$

is sharp, namely it is attained by any

$$f_j^* (x) = x_j^m, \;\; j = 1, \ldots, N, \;\; x = \left(x_1, \ldots, x_j, \ldots, x_N\right) \in \mathbb{R}^N . \tag{3.29}$$

Proof. We observe that (for $x, t \in \mathbb{R}^N$)

$$\begin{aligned} \Delta_t^m f_j^* (x) &= \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \left(x_j + i t_j\right)^m \\ &= m! t_j^m, \quad j = 1, \ldots, N. \end{aligned} \tag{3.30}$$

So that

$$\omega_m\left(f_j^*, h\right) = \sup_{\|t\|_2 \le h} m! \left|t_j\right|^m = m! \sup_{\|t\|_2 \le h} \left|t_j\right|^m = m! h^m. \tag{3.31}$$

Also

$$\Delta_t^m\left(\theta_n\left(f_j^*; x\right)\right) = \int_{\mathbb{R}^N}\left(\Delta_t^m f_j^*(x+s)\right) d\mu_{\xi_n}(s) \tag{3.32}$$

$$= \int_{\mathbb{R}^N} m! t_j^m d\mu_{\xi_n}(s) = m! t_j^m, \quad j = 1, \ldots, N, \text{ etc},$$

proving the claim. □

We need

Theorem 3.6. *Let* $f \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *a bounded sequence. Let* $\beta := (\beta_1, \ldots, \beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$; $|\beta| := \sum_{i=1}^N \beta_i = l$. *Here* $f(x + sj)$, $x, s \in \mathbb{R}^N$, *is* μ_{ξ_n}*-integrable wrt* s, *for* $j = 1, \ldots, r$. *There exist* μ_{ξ_n}*-integrable functions* $h_{i_1,j}$, $h_{\beta_1,i_2,j}$, $h_{\beta_1,\beta_2,i_3,j}, \ldots, h_{\beta_1,\beta_2,\ldots,\beta_{N-1},i_N,j} \ge 0$ *(*$j = 1, \ldots, r$*) on* $\mathbb{R}^N$ *such that*

$$\left|\frac{\partial^{i_1} f(x+sj)}{\partial x_1^{i_1}}\right| \le h_{i_1,j}(s), \quad i_1 = 1, \ldots, \beta_1, \tag{3.33}$$

$$\left|\frac{\partial^{\beta_1+i_2} f(x+sj)}{\partial x_2^{i_2}\partial x_1^{\beta_1}}\right| \le h_{\beta_1,i_2,j}(s), \quad i_2 = 1, \ldots, \beta_2,$$

$$\vdots$$

$$\left|\frac{\partial^{\beta_1+\beta_2+\ldots+\beta_{N-1}+i_N} f(x+sj)}{\partial x_N^{i_N}\partial x_{N-1}^{\beta_{N-1}}\ldots\partial x_2^{\beta_2}\partial x_1^{\beta_1}}\right| \le h_{\beta_1,\beta_2,\ldots,\beta_{N-1},i_N,j}(s), \quad i_N = 1, \ldots, \beta_N,$$

$\forall x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r,n}^{[\tilde{m}]}(f; x)\right)_\beta = \theta_{r,n}^{[\tilde{m}]}\left(f_\beta; x\right). \tag{3.34}$$

Proof. By H. Bauer [7], pp. 103–104. □

Corollary 3.7. *(to Theorem 3.6,* $r = 1$*) It holds*

$$(\theta_n(f; x))_\beta = \theta_n\left(f_\beta; x\right). \tag{3.35}$$

We present simultaneous global smoothness results.

Theorem 3.8. *Let $h > 0$ and assumptions of Theorem 3.6 are valid. Here $\gamma = 0, \beta$, $(0 = (0, \ldots, 0))$.*

i) Assume $\omega_m\left(f_\gamma, h\right) < \infty$. Then

$$\omega_m\left(\left(\theta_{r,n}^{[\tilde{m}]}(f)\right)_\gamma, h\right) \leq \left(\sum_{\tilde{j}=0}^{r}\left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right|\right)\omega_m\left(f_\gamma, h\right). \tag{3.36}$$

ii) Additionally suppose $f_\gamma \in L_1\left(\mathbb{R}^N\right)$. Then

$$\omega_m\left(\left(\theta_{r,n}^{[\tilde{m}]}(f)\right)_\gamma, h\right)_1 \leq \left(\sum_{\tilde{j}=0}^{r}\left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right|\right)\omega_m\left(f_\gamma, h\right)_1. \tag{3.37}$$

iii) Additionally suppose $f_\gamma \in L_p\left(\mathbb{R}^N\right)$, $p > 1$. Then

$$\omega_m\left(\left(\theta_{r,n}^{[\tilde{m}]}(f)\right)_\gamma, h\right)_p \leq \left(\sum_{\tilde{j}=0}^{r}\left|\alpha_{\tilde{j},r}^{[\tilde{m}]}\right|\right)\omega_m\left(f_\gamma, h\right)_p. \tag{3.38}$$

It follows

Corollary 3.9. *Let $h > 0$ and assumptions of Corollary 3.7 are valid. Here $\gamma = 0, \beta$.*

i) Assume $\omega_m\left(f_\gamma, h\right) < \infty$. Then

$$\omega_m\left(\left(\theta_n(f)\right)_\gamma, h\right) \leq \omega_m\left(f_\gamma, h\right). \tag{3.39}$$

ii) Additionally assume $f_\gamma \in L_1\left(\mathbb{R}^N\right)$. Then

$$\omega_m\left(\left(\theta_n(f)\right)_\gamma, h\right)_1 \leq \omega_m\left(f_\gamma, h\right)_1. \tag{3.40}$$

iii) Additionally assume $f_\gamma \in L_p\left(\mathbb{R}^N\right)$, $p > 1$. Then

$$\omega_m\left(\left(\theta_n(f)\right)_\gamma, h\right)_p \leq \omega_m\left(f_\gamma, h\right)_p. \tag{3.41}$$

Next comes multi-simultaneous approximation.

Theorem 3.10. *Let $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. The assumptions of Theorem 3.6 are valid. Call $\gamma = 0, \beta$. Assume $\left\|f_{\gamma+\alpha}\right\|_\infty < \infty$ and*

$$\int_{\mathbb{R}^N}\left(\prod_{i=1}^{N}|s_i|^{\alpha_i}\right)\left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \mathrm{d}\mu_{\xi_n}(s) < \infty,$$

for all $\alpha_j \in \mathbb{Z}^+$, $j = 1, \ldots, N$, $|\alpha| := \sum_{j=1}^{N} \alpha_j = m$, *where* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, *for* $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ *bounded sequence.*

For $\tilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \tilde{j}$, *call*

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, d\mu_{\xi_n}(s). \tag{3.42}$$

Then

$$\left\| \left(\theta_{r,n}^{[m]}(f; \cdot) \right)_\gamma - f_\gamma(\cdot) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1, \ldots, \alpha_N \geq 0: \\ |\alpha| = \tilde{j}}} \frac{c_{\alpha,n,\tilde{j}} f_{\gamma+\alpha}(\cdot)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_\infty \tag{3.43}$$

$$\leq \sum_{\left(\substack{\alpha_1, \ldots, \alpha_N \geq 0 \\ |\alpha| = m} \right)} \frac{\left(\omega_r \left(f_{\gamma+\alpha}, \xi_n \right) \right)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \left(\int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right).$$

Proof. Based on Theorems 6 and 9 of [2]. ☐

Theorem 3.11. *Let* $f \in C_B^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$ *(functions* l*-times continuously differentiable and bounded). The assumptions of Theorem 3.6 are valid. Call* $\gamma = 0, \beta$. *Suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{3.44}$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f \right)_\gamma - f_\gamma \right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \omega_r \left(f_\gamma, \xi_n \right). \tag{3.45}$$

Proof. By Theorems 6 and 11 of [2]. ☐

We continue with

Theorem 3.12. *Let* $f \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$. *The assumptions of Theorem 3.6 are valid. Call* $\gamma = 0, \beta$. *Let* $f_{(\gamma+\alpha)} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *for* $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ *bounded sequence. Assume for all* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = m$ *we have that*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{3.46}$$

For $\tilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \tilde{j}$, *call*

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, \mathrm{d}\mu_{\xi_n}(s) \,. \tag{3.47}$$

Then

$$\left\| \left(\theta_{r,n}^{[m]}(f;x)\right)_\gamma - f_\gamma(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_{\gamma+\alpha}(x)}{\prod_{i=1}^{N} \alpha_i!} \right) \right\|_{p,x}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \cdot \tag{3.48}$$

$$\left[\int_{\mathbb{R}^N} \left[\left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p \mathrm{d}\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \cdot \omega_r \left(f_{\gamma+\alpha}, \xi_n \right)_p \,.$$

Proof. By Theorems 6 and 4 of [3]. □

We give also

Theorem 3.13. *Let* $f \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. *The assumptions of Theorem 3.6 are valid. Call* $\gamma = 0, \beta$. *Let* $f_\gamma \in L_p\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{\mathrm{rp}} \mathrm{d}\mu_{\xi_n}(s) < \infty.$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f\right)_\gamma - f_\gamma \right\|_p \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{\mathrm{rp}} \mathrm{d}\mu_{\xi_n}(s) \right)^{\frac{1}{p}} \omega_r \left(f_\gamma, \xi_n \right)_p \,. \tag{3.49}$$

Proof. By Theorems 6 and 6 of [3]. □

Theorem 3.14. *Let* $f \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. *The assumptions of Theorem 3.6 are valid. Call* $\gamma = 0, \beta$. *Let* $f_\gamma \in L_1\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$. *Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also assume*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r \mathrm{d}\mu_{\xi_n}(s)<\infty.$$

Then

$$\left\|\left(\theta_{r,n}^{[0]}(f)\right)_\gamma - f_\gamma\right\|_1 \leq \left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^r \mathrm{d}\mu_{\xi_n}(s)\right)\omega_r\left(f_\gamma,\xi_n\right)_1. \tag{3.50}$$

Proof. By Theorems 6 and 8 of [3]. □

Theorem 3.15. *Let $f\in C^{m+l}\left(\mathbb{R}^N\right)$, $m,l,N\in\mathbb{N}$. The assumptions of Theorem 3.6 are valid. Call $\gamma=0,\beta$. Let $f_{(\gamma+\alpha)}\in L_1\left(\mathbb{R}^N\right)$, $|\alpha|=m$, $x\in\mathbb{R}^N$. Here, μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n>0$, $(\xi_n)_{n\in\mathbb{N}}$ is a bounded sequence. Suppose for all $\alpha:=(\alpha_1,\ldots,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,\ldots,N$, $|\alpha|:=\sum_{i=1}^N\alpha_i=m$, we have that*

$$\int_{\mathbb{R}^N}\left(\left(\prod_{i=1}^N|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right)\mathrm{d}\mu_{\xi_n}(s)<\infty. \tag{3.51}$$

For $\tilde{j}=1,\ldots,m$, and $\alpha:=(\alpha_1,\ldots,\alpha_N)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,\ldots,N$, $|\alpha|:=\sum_{i=1}^N\alpha_i=\tilde{j}$, call

$$c_{\alpha,n,\tilde{j}}:=\int_{\mathbb{R}^N}\prod_{i=1}^N s_i^{\alpha_i}\,\mathrm{d}\mu_{\xi_n}(s). \tag{3.52}$$

Then

$$\left\|\left(\theta_{r,n}^{[m]}(f;x)\right)_\gamma - f_\gamma(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]}\left(\sum_{|\alpha|=\tilde{j}}\frac{c_{\alpha,n,\tilde{j}}\,f_{(\gamma+\alpha)}(x)}{\prod_{i=1}^N\alpha_i!}\right)\right\|_{1,x} \tag{3.53}$$

$$\leq \sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^N\alpha_i!}\right)\omega_r\left(f_{\gamma+\alpha},\xi_n\right)_1\int_{\mathbb{R}^N}\left(\prod_{i=1}^N|s_i|^{\alpha_i}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\mathrm{d}\mu_{\xi_n}(s).$$

Proof. Based on Theorem 6 and Theorem 10 of [3]. □

3.3 Applications

Let all entities as in Sect. 3.2. We define the following specific operators:

(a) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}(f;x_1,\dots,x_N) := \frac{1}{(2\xi_n)^N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.54}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1j, x_2+s_2j,\dots,x_N+s_Nj)\, e^{-\frac{\left(\sum_{i=1}^{N}|s_i|\right)}{\xi_n}}\, ds_1\dots ds_N.$$

(b) The general multivariate Gauss–Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f;x_1,\dots,x_N) := \frac{1}{\left(\sqrt{\pi\xi_n}\right)^N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.55}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1j, x_2+s_2j,\dots,x_N+s_Nj)\, e^{-\frac{\left(\sum_{i=1}^{N}s_i^2\right)}{\xi_n}}\, ds_1\dots ds_N.$$

(c) The general multivariate Poisson–Cauchy singular integral operators:

$$U_{r,n}^{[m]}(f;x_1,\dots,x_N) := W_n^N\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.56}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1j,\dots,x_N+s_Nj)\prod_{i=1}^{N}\frac{1}{\left(s_i^{2\alpha}+\xi_n^{2\alpha}\right)^{\beta}}\, ds_1\dots ds_N,$$

with $\alpha\in\mathbb{N}$, $\beta>\frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta)\,\alpha\xi_n^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}. \tag{3.57}$$

(d) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}(f;x_1,\dots,x_N) := \lambda_n^{-N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \tag{3.58}$$

$$\int_{\mathbb{R}^N} f(x_1+s_1j,\dots,x_N+s_Nj)\prod_{i=1}^{N}\left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i}\right)^{2\beta} ds_1\dots ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta}\pi\,(-1)^{\beta}\,\beta\sum_{k=1}^{\beta}(-1)^k\,\frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}. \tag{3.59}$$

One can apply the results of this chapter to the operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and obtain interesting results.

3.4 Conclusion

The approximation results here imply important convergence properties of operators $\theta_{r,n}^{[m]}$ to the unit operator.

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Chapter 4
Multivariate Voronovskaya Asymptotic Expansions for General Singular Integrals

In this chapter, we continue with the study of approximation properties of smooth general singular integral operators over R^N, $N \geq 1$. We present multivariate Voronovskaya asymptotic type results and give quantitative results regarding the rate of convergence of multivariate singular integral operators to unit operator. We list specific multivariate singular integral operators that fulfill this theory. This chapter is based on [2].

4.1 Introduction

The main motivation for this chapter comes from [3–5]. We give here multivariate Voronovskaya-type asymptotic expansions regarding the multivariate singular integral operators, see Theorem 4.2 and Corollaries 4.3, 4.4. In Theorem 4.6, we present the simultaneous corresponding Voronovskaya asymptotic expansion for these operators. The expansions give also the rate of convergence of multivariate general singular integral operators to unit operator. In Sect. 4.3, we list the multivariate singular Picard, Gauss–Weierstrass, Poisson–Cauchy and Trigonometric operators that fulfill the main results.

4.2 Main Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \ldots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-m}, & \text{if } j = 0, \end{cases} \tag{4.1}$$

G.A. Anastassiou, *Approximation by Multivariate Singular Integrals*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4614-0589-4_4,

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \ldots, m \in \mathbb{N}. \tag{4.2}$$

Observe that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{4.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{4.4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the multiple smooth singular integral operators

$$\theta_{r,n}^{[m]} (f; x_1, \ldots, x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f (x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) \times d\mu_{\xi_n} (s), \tag{4.5}$$

where $s := (s_1, \ldots, s_N)$, $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}, m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

The above $\theta_{r,n}^{[m]}$ are not in general positive operators and they preserve constants, see [1].

We make

Remark 4.1. Here $f \in C^m \left(\mathbb{R}^N\right)$, $m, N \in \mathbb{N}$. Let $l = 0, 1, \ldots, m$. The lth order partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$ and $|\alpha| := \sum_{i=1}^N \alpha_i = l$.

Consider $g_z (t) := f (x_0 + t (z - x_0))$, $t \geq 0$; $x_0, z \in \mathbb{R}^N$.

Then

$$g_z^{(j)} (t) = \left[\left(\sum_{i=1}^{N} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f \right] (x_{01} + t (z_1 - x_{01}), \ldots, x_{0N} + t(z_N - x_{0N})), \tag{4.6}$$

for all $j = 0, 1, \ldots, m$.

In particular, we choose

$$z = (z_1, \ldots, z_N) = (x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) = x + sj,$$

and

$$x_0 = (x_{01}, \ldots, x_{0N}) = (x_1, x_2, \ldots, x_N) = x,$$

to get $g_{x+sj}(t) := f(x + t(sj))$.

Notice $g_{x+sj}(0) = f(x)$.

Also for $\tilde{j} = 0, 1, \ldots, m-1$ we get

$$g_{x+sj}^{(\tilde{j})}(0) = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i = \tilde{j}}} \left(\frac{\tilde{j}!}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (s_i j)^{\alpha_i} \right) f_\alpha(x). \tag{4.7}$$

Furthermore, we obtain

$$\frac{g_{x+sj}^{(m)}(\theta)}{m!} = \sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i = m}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \left(\prod_{i=1}^{N} (s_i j)^{\alpha_i} \right) f_\alpha(x + \theta(sj)), \tag{4.8}$$

$0 \leq \theta \leq 1$.

For $\tilde{j} = 1, \ldots, m-1$, and $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$, $|\alpha| := \sum_{i=1}^{N} \alpha_i = \tilde{j}$, we define

$$c_{\alpha,n,\tilde{j}} := c_{\alpha,n} := \int_{\mathbb{R}^N} \prod_{i=1}^{N} s_i^{\alpha_i} \, d\mu_{\xi_n}(s_1, \ldots, s_N). \tag{4.9}$$

Consequently, we derive

$$\sum_{\tilde{j}=1}^{m} \frac{\int_{\mathbb{R}^N} g_{x+sj}^{(\tilde{j})}(0) \, d\mu_{\xi_n}(s)}{\tilde{j}!}$$

$$= \sum_{\tilde{j}=1}^{m-1} j^{\tilde{j}} \left(\sum_{\substack{\alpha:=(\alpha_1,\ldots,\alpha_N),\ \alpha_i \in \mathbb{Z}^+, \\ i=1,\ldots,N,\ |\alpha|:=\sum_{i=1}^{N}\alpha_i = \tilde{j}}} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) c_{\alpha,n} f_\alpha(x) \right). \tag{4.10}$$

Next, we observe by multivariate Taylor's formula that

$$f\left(x+js\right)=g_{x+js}\left(1\right)=\sum_{\tilde{j}=0}^{m-1}\frac{g_{x+js}^{\left(\tilde{j}\right)}\left(0\right)}{\tilde{j}!}+\frac{g_{x+js}^{\left(m\right)}\left(\theta\right)}{m!},\tag{4.11}$$

where $\theta \in (0,1)$, which leads to

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(f\left(x+sj\right)-f\left(x\right)\right)\mathrm{d}\mu_{\xi_n}\left(s\right) \\ &=\sum_{\tilde{j}=1}^{m-1}j^{\tilde{j}}\left(\sum_{|\alpha|=\tilde{j}}\frac{1}{\left(\prod_{i=1}^{N}\alpha_i!\right)}c_{\alpha,n,\tilde{j}}f_{\alpha}\left(x\right)\right) \\ &\quad+j^m\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i}f_{\alpha}\left(x+\theta sj\right)\mathrm{d}\mu_{\xi_n}\left(s\right).\end{aligned}\tag{4.12}$$

Hence

$$\begin{aligned}\theta_{r,n}^{[m]}\left(f;x\right)-f\left(x\right)&=\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\int_{\mathbb{R}^N}\left(f\left(x+sj\right)-f\left(x\right)\right)\mathrm{d}\mu_{\xi_n}\left(s\right) \\ &=\sum_{\tilde{j}=1}^{m-1}\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^{\tilde{j}}\left(\sum_{|\alpha|=\tilde{j}}\frac{1}{\left(\prod_{i=1}^{N}\alpha_i!\right)}c_{\alpha,n,\tilde{j}}f_{\alpha}\left(x\right)\right) \qquad (4.13)\\ &\quad+\sum_{j=1}^{r}\alpha_{j,r}^{[m]}j^m\sum_{|\alpha|=m}\left(\frac{1}{\prod_{i=1}^{N}\alpha_i!}\right)\int_{\mathbb{R}^N}\prod_{i=1}^{N}s_i^{\alpha_i} \\ &\quad\times f_{\alpha}\left(x+\theta sj\right)\mathrm{d}\mu_{\xi_n}\left(s\right)\end{aligned}$$

$$= \sum_{\tilde{j}=1}^{m-1} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_{\alpha}(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \tag{4.14}$$

$$+ \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right)$$

$$\times \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f_{\alpha}(x + \theta s j) \right) d\mu_{\xi_n}(s).$$

Thus, we have

$$\psi := \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^{m-1} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_{\alpha}(x)}{\left(\prod_{i=1}^{N} \alpha_i! \right)} \right) \tag{4.15}$$

$$= \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) \left(\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f_{\alpha}(x + \theta s j) \right) d\mu_{\xi_n}(s). \tag{4.16}$$

Call

$$\phi_{\alpha}(x, s) := \sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} f_{\alpha}(x + \theta s j). \tag{4.17}$$

Thus,

$$\psi = \sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i!} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i} \right) \phi_{\alpha}(x, s)\, d\mu_{\xi_n}(s). \tag{4.18}$$

Consider

$$\Delta_{\xi_n} := \frac{\psi}{\xi_n^m}. \tag{4.19}$$

Suppose f_α is bounded for all $\alpha : |\alpha| = m$, by $M > 0$. That is $\| f_\alpha \|_\infty \leq M$. Therefore,

$$|\phi_\alpha (x, s)| \leq \left(\sum_{j=1}^{r} \binom{r}{j} \right) M = (2^r - 1) M. \tag{4.20}$$

Consequently,

$$\left| \Delta_{\xi_n} \right| \leq \frac{(2^r - 1) M}{\xi_n^m} \left(\sum_{|\alpha|=m} \left(\frac{1}{\prod_{i=1}^{N} \alpha_i !} \right) \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n} (s) \right). \tag{4.21}$$

Suppose for $|\alpha| = m$ that

$$\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i} \right) d\mu_{\xi_n} (s) \leq \rho, \text{ for any } (\xi_n)_{n \in \mathbb{N}} . \tag{4.22}$$

Therefore,

$$\left| \Delta_{\xi_n} \right| \leq (2^r - 1) M\rho \left(\sum_{|\alpha|=m} \frac{1}{\left(\prod_{i=1}^{N} \alpha_i ! \right)} \right) =: \lambda, \quad \lambda > 0. \tag{4.23}$$

Hence

$$\frac{|\psi|}{\xi_n^m} \leq \lambda \quad \text{and} \quad \frac{|\psi| \, \xi_n^\gamma}{\xi_n^m} \leq \lambda \xi_n^\gamma \to 0, \tag{4.24}$$

where $0 < \gamma \leq 1$, as $\xi_n \to 0+$.

That is

$$\frac{|\psi|}{\xi_n^{m-\gamma}} \to 0, \quad \text{as } \xi_n \to 0+, \tag{4.25}$$

which means $\psi = 0 \left(\xi_n^{m-\gamma} \right)$.

We established

Theorem 4.2. *Let* $f \in C^m(\mathbb{R}^N)$, $m, N \in \mathbb{N}$,*with all* $\|f_\alpha\|_\infty \le M$, $M > 0$, *all* $\alpha : |\alpha| = m$. *Let* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence,* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$.

Call $c_{\alpha,n,\tilde{j}} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^N s_i^{\alpha_i}\right) d\mu_{\xi_n}(s)$, *all* $|\alpha| = \tilde{j} = 1, \ldots, m-1$. *Suppose* $\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^N |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \le \rho$, *all* $\alpha : |\alpha| = m$, $\rho > 0$, *for any such* $(\xi_n)_{n\in\mathbb{N}}$. *Also* $0 < \gamma \le 1$, $x \in \mathbb{R}^N$. *Then*

$$\theta_{r,n}^{[m]}(f; x) - f(x) = \sum_{\tilde{j}=1}^{m-1} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\left(\prod_{i=1}^N \alpha_i!\right)} \right) + 0\left(\xi_n^{m-\gamma}\right). \tag{4.26}$$

When $m = 1$, *the sum collapses.*

Above we assume $\theta_{r,n}^{[m]}(f; x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$.

Corollary 4.3. *Let* $f \in C^1(\mathbb{R}^N)$, $N \ge 1$,*with all* $\left\|\frac{\partial f}{\partial x_i}\right\|_\infty \le M$, $M > 0$, $i = 1, \ldots, N$. *Let* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence,* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$. *Suppose*

$$\xi_n^{-1} \int_{\mathbb{R}^N} |s_i|\, d\mu_{\xi_n}(s) \le \rho, \quad \text{all } i = 1, \ldots, N, \tag{4.27}$$

$\rho > 0$, *for any such* $(\xi_n)_{n\in\mathbb{N}}$. *Also* $0 < \gamma \le 1$, $x \in \mathbb{R}^N$. *Then*

$$\theta_{r,n}^{[1]}(f; x) - f(x) = 0\left(\xi_n^{1-\gamma}\right). \tag{4.28}$$

Above we assume $\theta_{r,n}^{[1]}(f; x) \in \mathbb{R}$, $\forall\, x \in \mathbb{R}^N$.

Corollary 4.4. *Let* $f \in C^2(\mathbb{R}^2)$, *with all* $\left\|\frac{\partial^2 f}{\partial x_1^2}\right\|_\infty, \left\|\frac{\partial^2 f}{\partial x_2^2}\right\|_\infty, \left\|\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\|_\infty \le M$, $M > 0$. *Let* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence,* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^2$.

Call

$$c_1 = \int_{\mathbb{R}^2} s_1 d\mu_{\xi_n}(s), \quad c_2 = \int_{\mathbb{R}^2} s_2 d\mu_{\xi_n}(s). \tag{4.29}$$

Suppose

$$\xi_n^{-2} \int_{\mathbb{R}^2} s_1^2 d\mu_{\xi_n}(s), \;\; \xi_n^{-2} \int_{\mathbb{R}^2} s_2^2 d\mu_{\xi_n}(s), \;\; \xi_n^{-2} \int_{\mathbb{R}^2} |s_1|\,|s_2|\, d\mu_{\xi_n}(s) \le \rho,$$

$\rho > 0$, *for any such* $(\xi_n)_{n\in\mathbb{N}}$. *Also* $0 < \gamma \leq 1$, $x \in \mathbb{R}^2$. *Then*

$$\theta_{r,n}^{[2]}(f;x) - f(x) = \left(\sum_{j=1}^{r} \alpha_{j,r}^{[2]} j\right)\left(c_1 \frac{\partial f}{\partial x_1}(x) + c_2 \frac{\partial f}{\partial x_2}(x)\right) + 0\left(\xi_n^{2-\gamma}\right). \quad (4.30)$$

We continue with

Theorem 4.5. *Let* $f \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *a bounded sequence. Let* $\beta := (\beta_1, \ldots, \beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$; $|\beta| := \sum_{i=1}^{N} \beta_i = l$. *Here* $f(x + sj)$, $x, s \in \mathbb{R}^N$, *is* μ_{ξ_n}*-integrable wrt s, for* $j = 1, \ldots, r$. *There exist* μ_{ξ_n}*-integrable functions* $h_{i_1,j}$, $h_{\beta_1,i_2,j}, h_{\beta_1,\beta_2,i_3,j}, \ldots, h_{\beta_1,\beta_2,\ldots,\beta_{N-1},i_N,j} \geq 0$ *(*$j = 1, \ldots, r$*) on* $\mathbb{R}^N$ *such that*

$$\left|\frac{\partial^{i_1} f(x + sj)}{\partial x_1^{i_1}}\right| \leq h_{i_1,j}(s), \quad i_1 = 1, \ldots, \beta_1,$$

$$\left|\frac{\partial^{\beta_1 + i_2} f(x + sj)}{\partial x_2^{i_2} \partial x_1^{\beta_1}}\right| \leq h_{\beta_1,i_2,j}(s), \quad i_2 = 1, \ldots, \beta_2,$$

$$\vdots$$

$$\left|\frac{\partial^{\beta_1 + \beta_2 + \ldots + \beta_{N-1} + i_N} f(x + sj)}{\partial x_N^{i_N} \partial x_{N-1}^{\beta_{N-1}} \ldots \partial x_2^{\beta_2} \partial x_1^{\beta_1}}\right| \leq h_{\beta_1,\beta_2,\ldots,\beta_{N-1},i_N,j}(s), \quad i_N = 1, \ldots, \beta_N, \quad (4.31)$$

$\forall\, x, s \in \mathbb{R}^N$.

Then, both of the next exist and

$$\left(\theta_{r,n}^{[m]}(f;x)\right)_\beta = \theta_{r,n}^{[m]}\left(f_\beta; x\right). \quad (4.32)$$

Proof. By H. Bauer [6], pp. 103–104. □

We finish this chapter with

Theorem 4.6. *Let* $f \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. *Assumptions of Theorem 4.5 are valid. Call* $\gamma = 0, \beta$. *Suppose* $\left\| f_{\gamma+\alpha} \right\|_\infty \leq M$, $M > 0$, *for all* $\alpha : |\alpha| = m$. *Let* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *bounded sequence,* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$. *Call* $c_{\alpha,n,\tilde{j}} = \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} s_i^{\alpha_i}\right) d\mu_{\xi_n}(s)$, *all* $|\alpha| = \tilde{j} = 1, \ldots, m-1$. *Assume* $\xi_n^{-m} \int_{\mathbb{R}^N} \left(\prod_{i=1}^{N} |s_i|^{\alpha_i}\right) d\mu_{\xi_n}(s) \leq \rho$, *all* $\alpha : |\alpha| = m$, $\rho > 0$, *for any such* $(\xi_n)_{n\in\mathbb{N}}$. *Also* $0 < \gamma \leq 1$, $x \in \mathbb{R}^N$. *Then*

$$\left(\theta_{r,n}^{[m]}\left(f;x\right)\right)_{\gamma}-f_{\gamma}\left(x\right)=\sum_{\tilde{j}=1}^{m-1}\delta_{\tilde{j},r}^{[m]}\left(\sum_{|\alpha|=\tilde{j}}\frac{c_{\alpha,n,\tilde{j}}\,f_{\gamma+\alpha}\left(x\right)}{\left(\prod_{i=1}^{N}\alpha_{i}!\right)}\right)+0\left(\xi_{n}^{m-\gamma}\right). \quad (4.33)$$

When $m=1$, the sum collapses.

4.3 Applications

Let all entities as in Sect. 4.2. We define the following specific operators:

(a) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}\left(f;x_{1},\ldots,x_{N}\right):=\frac{1}{\left(2\xi_{n}\right)^{N}}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \quad (4.34)$$

$$\int_{\mathbb{R}^{N}}f\left(x_{1}+s_{1}j,x_{2}+s_{2}j,\ldots,x_{N}+s_{N}j\right)\mathrm{e}^{-\frac{\left(\sum_{i=1}^{N}|s_{i}|\right)}{\xi_{n}}}\mathrm{d}s_{1}\ldots\mathrm{d}s_{N}.$$

(b) The general multivariate Gauss–Weierstrass singular integral operators:

$$W_{r,n}^{[m]}\left(f;x_{1},\ldots,x_{N}\right):=\frac{1}{\left(\sqrt{\pi\xi_{n}}\right)^{N}}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \quad (4.35)$$

$$\int_{\mathbb{R}^{N}}f\left(x_{1}+s_{1}j,x_{2}+s_{2}j,\ldots,x_{N}+s_{N}j\right)\mathrm{e}^{-\frac{\left(\sum_{i=1}^{N}s_{i}^{2}\right)}{\xi_{n}}}\mathrm{d}s_{1}\ldots\mathrm{d}s_{N}.$$

(c) The general multivariate Poisson–Cauchy singular integral operators:

$$U_{r,n}^{[m]}\left(f;x_{1},\ldots,x_{N}\right):=W_{n}^{N}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\cdot \quad (4.36)$$

$$\int_{\mathbb{R}^{N}}f\left(x_{1}+s_{1}j,\ldots,x_{N}+s_{N}j\right)\prod_{i=1}^{N}\frac{1}{\left(s_{i}^{2\alpha}+\xi_{n}^{2\alpha}\right)^{\beta}}\mathrm{d}s_{1}\ldots\mathrm{d}s_{N},$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta)\, \alpha \xi_n^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}. \tag{4.37}$$

(d) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{4.38}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, \ldots, x_N + s_N j) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} ds_1 \ldots ds_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta} \pi (-1)^{\beta} \beta \sum_{k=1}^{\beta} (-1)^k \frac{k^{2\beta-1}}{(\beta-k)!\,(\beta+k)!}. \tag{4.39}$$

One can apply the results of this chapter to the operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and obtain interesting results.

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Chapter 5
Simultaneous Approximation by Multivariate Complex General Singular Integrals

Here, we present complex multivariate simultaneous approximation for general smooth singular integral operators converging with rates to the unit operator. The associated and presented inequalities are in $\|\cdot\|_p$, $1 \leq p \leq \infty$ norm and they involve multivariate related moduli of smoothness. At the end, we list as this theory's applicators the special cases of multivariate complex Picard, Gauss–Weierstrass, Poisson–Cauchy and Trigonometric singular integral operators. This chapter relies on [2].

5.1 Introduction

Here, we are motivated by [1, 3, 4] and expand these works to complex valued functions. We present simultaneous approximation in $\|\cdot\|_p$, $1 \leq p \leq \infty$, of multivariate general smooth singular integral operators to the unit operator with rates. At the end, we list specific operators where our theory can be applied. From our approximation results, one can derive interesting convergence properties of these general operators. The expansion to complex case is based on basic properties of complex numbers and complex valued functions.

5.2 Main Results

Here $r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, we define

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m}, & \text{if } j = 1, 2, \ldots, r, \\ 1 - \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} i^{-m}, & \text{if } j = 0, \end{cases} \tag{5.1}$$

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and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, \ldots, m \in \mathbb{N}. \tag{5.2}$$

See that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1, \tag{5.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{5.4}$$

Let μ_{ξ_n} be a probability Borel measure on $\mathbb{R}^N$, $N \geq 1$, $\xi_n > 0$, $n \in \mathbb{N}$.

We now define the real multiple smooth singular integral operators

$$\theta_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j) \, d\mu_{\xi_n}(s), \tag{5.5}$$

where $s := (s_1, \ldots, s_N)$, $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$; $n, r \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $f : \mathbb{R}^N \to \mathbb{R}$ is a Borel measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

Above operators $\theta_{r,n}^{[m]}$ are not in general positive operators and they preserve constants, see [3].

Definition 5.1. Let $f \in C(\mathbb{R}^N)$, $N \geq 1$, $m \in \mathbb{N}$, the mth modulus of smoothness for $1 \leq p \leq \infty$, is given by

$$\omega_m(f; h)_p := \sup_{\|t\|_2 \leq h} \|\Delta_t^m f(x)\|_{p,x}, \tag{5.6}$$

$h > 0$, where

$$\Delta_t^m f(x) := \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x + jt). \tag{5.7}$$

Denote

$$\omega_m(f; h)_\infty = \omega_m(f, h). \tag{5.8}$$

Above, $x, t \in \mathbb{R}^N$.

We make

Remark 5.2. We consider here complex valued Borel measurable functions $f : \mathbb{R}^N \to \mathbb{C}$ such that $f = f_1 + i f_2$, $i = \sqrt{-1}$, where $f_1, f_2 : \mathbb{R}^N \to \mathbb{R}$ are implied to be real valued Borel measurable functions.

We define the complex singular operators

$$\theta_{r,n}^{[m]}(f;x) := \theta_{r,n}^{[m]}(f_1;x) + i\,\theta_{r,n}^{[m]}(f_2;x)\,, \quad x \in \mathbb{R}^N. \tag{5.9}$$

We assume that $\theta_{r,n}^{[m]}\left(f_j;x\right) \in \mathbb{R}$, $\forall\ x \in \mathbb{R}^N$, $j = 1,2$.

One notices easily that

$$\left|\theta_{r,n}^{[m]}(f;x) - f(x)\right| \tag{5.10}$$
$$\leq \left|\theta_{r,n}^{[m]}(f_1;x) - f_1(x)\right| + \left|\theta_{r,n}^{[m]}(f_2;x) - f_2(x)\right|$$

also

$$\left\|\theta_{r,n}^{[m]}(f;x) - f(x)\right\|_{\infty,x} \tag{5.11}$$
$$\leq \left\|\theta_{r,n}^{[m]}(f_1;x) - f_1(x)\right\|_{\infty,x} + \left\|\theta_{r,n}^{[m]}(f_2;x) - f_2(x)\right\|_{\infty,x}$$

and

$$\left\|\theta_{r,n}^{[m]}(f) - f\right\|_p \tag{5.12}$$
$$\leq \left\|\theta_{r,n}^{[m]}(f_1) - f_1\right\|_p + \left\|\theta_{r,n}^{[m]}(f_2) - f_2\right\|_p, \quad p \geq 1.$$

Furthermore, it holds

$$f_\alpha(x) = f_{1,\alpha}(x) + i f_{2,\alpha}(x)\,, \tag{5.13}$$

where α denotes a partial derivative of any order and arrangement.

Here based on Theorem 9 of [3], we obtain

Theorem 5.3. *Let $f : \mathbb{R}^N \to \mathbb{C}$, $N \geq 1$, such that $f = f_1 + if_2$, $j = 1,2$. Here $m \in \mathbb{N}$, $f_j \in C^m\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$. Assume $\left\|f_{j,\alpha}\right\|_\infty < \infty$, for all $\alpha_k \in \mathbb{Z}^+$, $k = 1,\ldots,N : |\alpha| = \sum_{k=1}^N \alpha_k = m$; $j = 1,2$. Let μ_{ξ_n} be a Borel probability measure on $\mathbb{R}^N$, for $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ bounded sequence. Assume that for all $\alpha := (\alpha_1,\ldots,\alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1,\ldots,N$, $|\alpha| := \sum_{k=1}^N \alpha_k = m$ we have that*

$$\int_{\mathbb{R}^N} \left(\prod_{k=1}^N |s_k|^{\alpha_k}\right)\left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{5.14}$$

For $\tilde{j} = 1,\ldots,m$, and $\alpha := (\alpha_1,\ldots,\alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1,\ldots,N$, $|\alpha| := \sum_{k=1}^N \alpha_k = \tilde{j}$, call

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^N s_k^{\alpha_k} \mathrm{d}\mu_{\xi_n}(s_1,\ldots,s_N)\,. \tag{5.15}$$

Then

$$\left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{\substack{\alpha_1,\ldots,\alpha_N \geq 0: \\ |\alpha|=\tilde{j}}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_{\infty,x}$$

$$\leq \sum_{\left(\substack{\alpha_1,\ldots,\alpha_N \geq 0 \\ |\alpha|=m}\right)} \frac{(\omega_r(f_{1,\alpha},\xi_n) + \omega_r(f_{2,\alpha},\xi_n))}{\left(\prod_{k=1}^{N} \alpha_k!\right)} \cdot \tag{5.16}$$

$$\left(\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right), \quad x \in \mathbb{R}^N.$$

The $m = 0$ case follows

Corollary 5.4. *Let* $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $N \geq 1$. *Here* $j = 1, 2$. *Let* $f_j \in C_B(\mathbb{R}^N)$ *(continuous and bounded functions). Then*

$$\left\| \theta_{r,n}^{[0]} f - f \right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \cdot \tag{5.17}$$

$$(\omega_r(f_1,\xi_n) + \omega_r(f_2,\xi_n)),$$

by assuming

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty. \tag{5.18}$$

Proof. By Theorem 11 of [3]. □

Theorem 5.5 ([1]). *Let* $f \in C^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$, $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *a bounded sequence. Let* $\beta := (\beta_1, \ldots, \beta_N)$, $\beta_i \in \mathbb{Z}^+$, $i = 1, \ldots, N$; $|\beta| := \sum_{i=1}^{N} \beta_i = l$. *Here,* $f(x + sj)$, $x, s \in \mathbb{R}^N$, *is* μ_{ξ_n}*-integrable wrt* s, *for* $j = 1, \ldots, r$. *There exist* μ_{ξ_n}*-integrable functions* $h_{i_1,j}$, $h_{\beta_1,i_2,j}$, $h_{\beta_1,\beta_2,i_3,j}, \ldots, h_{\beta_1,\beta_2,\ldots,\beta_{N-1},i_N,j} \geq 0$ *(*$j = 1, \ldots, r$*) on* $\mathbb{R}^N$ *such that*

$$\left| \frac{\partial^{i_1} f(x + sj)}{\partial x_1^{i_1}} \right| \leq h_{i_1,j}(s), \quad i_1 = 1, \ldots, \beta_1, \tag{5.19}$$

$$\left| \frac{\partial^{\beta_1+i_2} f(x + sj)}{\partial x_2^{i_2} \partial x_1^{\beta_1}} \right| \leq h_{\beta_1,i_2,j}(s), \quad i_2 = 1, \ldots, \beta_2,$$

$$\vdots$$

$$\left|\frac{\partial^{\beta_1+\beta_2+\ldots+\beta_{N-1}+i_N} f\,(x+sj)}{\partial x_N^{i_N}\,\partial x_{N-1}^{\beta_{N-1}}\ldots\partial x_2^{\beta_2}\,\partial x_1^{\beta_1}}\right| \leq h_{\beta_1,\beta_2,\ldots,\beta_{N-1},i_N,j}\,(s)\,,\quad i_N=1,\ldots,\beta_N, \tag{5.1}$$

$\forall\ x, s \in \mathbb{R}^N$.

Then, bothof the next exist and

$$\left(\theta_{r,n}^{[\tilde{m}]}\,(f;x)\right)_\beta = \theta_{r,n}^{[\tilde{m}]}\left(f_\beta;x\right). \tag{5.20}$$

We give

Theorem 5.6. *Let $f:\mathbb{R}^N\to\mathbb{C}$ such that $f=f_1+if_2$. Here $j=1,2$. Let $f_j\in C^{m+l}\left(\mathbb{R}^N\right)$, $m,l,N\in\mathbb{N}$. For f_j, the assumptions of Theorem 5.5 are valid. Call $\gamma=0,\beta$. Assume $\left\|f_{(\gamma+\alpha)}\right\|_\infty<\infty$ and*

$$\int_{\mathbb{R}^N}\left(\prod_{k=1}^N|s_k|^{\alpha_k}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r \mathrm{d}\mu_{\xi_n}\,(s)<\infty, \tag{5.21}$$

where μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$, for $\xi_n>0$, $(\xi_n)_{n\in\mathbb{N}}$ is bounded sequence; for all $\alpha_k\in\mathbb{Z}^+$, $k=1,\ldots,N$: $|\alpha|=\sum_{k=1}^N\alpha_k=m$.

For $\tilde{j}=1,\ldots,m$, and $\alpha:=(\alpha_1,\ldots,\alpha_N)$, $\alpha_k\in\mathbb{Z}^+$, $k=1,\ldots,N$, $|\alpha|:=\sum_{k=1}^N\alpha_k=\tilde{j}$, call

$$c_{\alpha,n,\tilde{j}}:=\int_{\mathbb{R}^N}\prod_{k=1}^N s_k^{\alpha_k}\,\mathrm{d}\mu_{\xi_n}\,(s)\,.$$

Then

$$\left\|\left(\theta_{r,n}^{[m]}\,(f;\cdot)\right)_\gamma - f_\gamma\,(\cdot)-\sum_{\tilde{j}=1}^m\delta_{\tilde{j},r}^{[m]}\left(\sum_{\substack{\alpha_1,\ldots,\alpha_N\geq 0:\\|\alpha|=\tilde{j}}}\frac{c_{\alpha,n,\tilde{j}}\,f_{\gamma+\alpha}\,(\cdot)}{\prod_{k=1}^N\alpha_k!}\right)\right\|_\infty$$

$$\leq\sum_{\left(\substack{\alpha_1,\ldots,\alpha_N\geq 0\\|\alpha|=m}\right)}\frac{\left(\omega_r\left(f_{1,\gamma+\alpha},\xi_n\right)+\omega_r\left(f_{2,\gamma+\alpha},\xi_n\right)\right)}{\left(\prod_{k=1}^N\alpha_k!\right)}\cdot$$

$$\left(\int_{\mathbb{R}^N}\left(\prod_{k=1}^N|s_k|^{\alpha_k}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\mathrm{d}\mu_{\xi_n}\,(s)\right). \tag{5.22}$$

Proof. By Theorem 10 of [1]. ☐

Also, we have

Theorem 5.7. *Let $f : \mathbb{R}^N \to \mathbb{C}$ such that $f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C_B^l(\mathbb{R}^N)$, $l, N \in \mathbb{N}$. The assumptions of Theorem 5.5 are valid for f_j. Call $\gamma = 0, \beta$. Assume*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \mathrm{d}\mu_{\xi_n}(s) < \infty.$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f\right)_\gamma - f_\gamma \right\|_\infty \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \mathrm{d}\mu_{\xi_n}(s) \right) \cdot \tag{5.23}$$
$$\left(\omega_r\left(f_{1,\gamma}, \xi_n\right) + \omega_r\left(f_{2,\gamma}, \xi_n\right)\right).$$

Proof. By Theorem 11 of [1]. □

By Theorem 4 of [4], we get

Theorem 5.8. *Let $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$. Here $j = 1, 2$. Let $f_j \in C^m(\mathbb{R}^N)$, $m \in \mathbb{N}$, $N \geq 1$, with $f_{j,\alpha} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. Let $p, q > 1$: $\frac{1}{p} + \frac{1}{q} = 1$. Here μ_{ξ_n} is a Borel probability measure on $\mathbb{R}^N$ for $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence. Assume for all $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^N \alpha_k = m$, we have that*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{k=1}^N |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n}\right)^r \right)^p \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{5.24}$$

For $\tilde{j} = 1, \ldots, m$, and $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^N \alpha_k = \tilde{j}$, call

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^N s_k^{\alpha_k} \mathrm{d}\mu_{\xi_n}(s). \tag{5.25}$$

Then

$$\left\| \theta_{r,n}^{[m]}(f; x) - f(x) - \sum_{\tilde{j}=1}^m \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha| = \tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\prod_{k=1}^N \alpha_k!} \right) \right\|_{p,x}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha| = m} \frac{1}{\left(\prod_{k=1}^N \alpha_k! \right)} \right) \cdot$$

$$\left[\int_{\mathbb{R}^N}\left[\left(\prod_{k=1}^{N}|s_k|^{\alpha_k}\right)\left(1+\frac{\|s\|_2}{\xi_n}\right)^r\right]^p \mathrm{d}\mu_{\xi_n}(s)\right]^{\frac{1}{p}} \cdot$$

$$\left(\omega_r\left(f_{1,\alpha},\xi_n\right)_p+\omega_r\left(f_{2,\alpha},\xi_n\right)_p\right). \tag{5.26}$$

We further get

Theorem 5.9. *Let* $f : \mathbb{R}^N \to \mathbb{C} : f = f_1 + if_2$, $j = 1, 2$. *Let* $f_j \in \left(C\left(\mathbb{R}^N\right) \cap L_p\left(\mathbb{R}^N\right)\right)$, $N \geq 1$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{\mathrm{rp}} \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{5.27}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_p \leq \left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{\mathrm{rp}} \mathrm{d}\mu_{\xi_n}(s)\right)^{\frac{1}{p}} \cdot \tag{5.28}$$

$$\left(\omega_r\left(f_1,\xi_n\right)_p+\omega_r\left(f_2,\xi_n\right)_p\right).$$

Proof. By Theorem 6 of [4]. □

Based on Theorem 8 of [4], we get

Theorem 5.10. *Let* $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + if_2$, $j = 1, 2$. *Let* $f_j \in \left(C\left(\mathbb{R}^N\right) \cap L_1\left(\mathbb{R}^N\right)\right)$, $N \geq 1$. *Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{r} \mathrm{d}\mu_{\xi_n}(s) < \infty. \tag{5.29}$$

Then

$$\left\|\theta_{r,n}^{[0]}(f)-f\right\|_1 \leq \left(\int_{\mathbb{R}^N}\left(1+\frac{\|s\|_2}{\xi_n}\right)^{r} \mathrm{d}\mu_{\xi_n}(s)\right) \cdot \tag{5.30}$$

$$\left(\omega_r\left(f_1,\xi_n\right)_1+\omega_r\left(f_2,\xi_n\right)_1\right).$$

Based on Theorem 10 of [4], we get

Theorem 5.11. *Let* $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + if_2$, $j = 1, 2$. *Let* $f_j \in C^m\left(\mathbb{R}^N\right)$, $m, N \in \mathbb{N}$, *with* $f_{j,\alpha} \in L_1\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *for* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *is a bounded sequence. Assume for all* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = m$ *that we have*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right) d\mu_{\xi_n}(s) < \infty. \tag{5.31}$$

For $\tilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = \tilde{j}$, *call*

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^{N} s_k^{\alpha_k} d\mu_{\xi_n}(s). \tag{5.32}$$

Then

$$\left\| \theta_{r,n}^{[m]}(f;x) - f(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_\alpha(x)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_{1,x}$$

$$\leq \sum_{|\alpha|=m} \left(\frac{1}{\prod_{k=1}^{N} \alpha_k!} \right) \left(\omega_r(f_{1,\alpha}, \xi_n)_1 + \omega_r(f_{2,\alpha}, \xi_n)_1 \right) \cdot \tag{5.33}$$

$$\int_{\mathbb{R}^N} \left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s). \tag{5.2}$$

Basedon Theorem 12 of [1], we get

Theorem 5.12. *Let* $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + i f_2$; $j = 1, 2$, *with* $f_j \in C^{m+l}(\mathbb{R}^N)$, $m, l, N \in \mathbb{N}$. *The assumptions of Theorem 5.5 are valid for* f_j. *Call* $\gamma = 0, \beta$. *Let* $f_{j,(\gamma+\alpha)} \in L_p(\mathbb{R}^N)$, $|\alpha| = m$, $x \in \mathbb{R}^N$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here,* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *for* $\xi_n > 0$, $(\xi_n)_{n \in \mathbb{N}}$ *is a bounded sequence. Assume for all* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = m$ *we have that*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right)^p d\mu_{\xi_n}(s) < \infty. \tag{5.34}$$

For $\tilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^{N} \alpha_k = \tilde{j}$, *call*

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^{N} s_k^{\alpha_k} d\mu_{\xi_n}(s).$$

Then

$$\left\| \left(\theta_{r,n}^{[m]}(f;x)\right)_{\gamma} - f_{\gamma}(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_{\gamma+\alpha}(x)}{\prod_{k=1}^{N} \alpha_k!} \right) \right\|_{p,x} \tag{5.35}$$

$$\leq \left(\frac{m}{(q(m-1)+1)^{\frac{1}{q}}} \right) \left(\sum_{|\alpha|=m} \frac{1}{\left(\prod_{k=1}^{N} \alpha_k! \right)} \right) \cdot$$

$$\left[\int_{\mathbb{R}^N} \left[\left(\prod_{k=1}^{N} |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right]^p d\mu_{\xi_n}(s) \right]^{\frac{1}{p}} \cdot$$

$$\left(\omega_r\left(f_{1,\gamma+\alpha}, \xi_n\right)_p + \omega_r\left(f_{2,\gamma+\alpha}, \xi_n\right)_p \right).$$

Based on Theorem 13 of [1] we get

Theorem 5.13. *Let* $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + if_2$, $j = 1, 2$. *Let* $f_j \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. *The assumptions of Theorem 5.5 are valid. Call* $\gamma = 0, \beta$. *Let* $f_{j,\gamma} \in L_p\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{\mathrm{rp}} d\mu_{\xi_n}(s) < \infty.$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f\right)_{\gamma} - f_{\gamma} \right\|_p \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^{\mathrm{rp}} d\mu_{\xi_n}(s) \right)^{\frac{1}{p}} \cdot \tag{5.36}$$

$$\left(\omega_r\left(f_{1,\gamma}, \xi_n\right)_p + \omega_r\left(f_{2,\gamma}, \xi_n\right)_p \right).$$

By Theorem 14 of [1], we get

Theorem 5.14. *Let* $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + if_2$, $j = 1, 2$. *Let* $f_j \in C^l\left(\mathbb{R}^N\right)$, $l, N \in \mathbb{N}$. *The assumptions of Theorem 5.5 are valid for* f_j. *Call* $\gamma = 0, \beta$. *Let* $f_{j,\gamma} \in L_1\left(\mathbb{R}^N\right)$, $x \in \mathbb{R}^N$. *Assume* μ_{ξ_n} *probability Borel measures on* $\mathbb{R}^N$, $(\xi_n)_{n\in\mathbb{N}} > 0$ *and bounded. Also suppose*

$$\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) < \infty.$$

Then

$$\left\| \left(\theta_{r,n}^{[0]} f \right)_{\gamma} - f_{\gamma} \right\|_{1} \leq \left(\int_{\mathbb{R}^N} \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s) \right) \cdot \left(\omega_r \left(f_{1,\gamma}, \xi_n \right)_1 + \omega_r \left(f_{2,\gamma}, \xi_n \right)_1 \right). \tag{5.37}$$

Finally, we have

Theorem 5.15. *Let* $f : \mathbb{R}^N \to \mathbb{C}$, $f = f_1 + i f_2$, $j = 1, 2$, *with* $f_j \in C^{m+l}\left(\mathbb{R}^N\right)$, $m, l, N \in \mathbb{N}$. *The assumptions of Theorem 5.5 are valid for* f_j. *Call* $\gamma = 0, \beta$. *Let* $f_{j,(\gamma+\alpha)} \in L_1\left(\mathbb{R}^N\right)$, $|\alpha| = m$, $x \in \mathbb{R}^N$. *Here* μ_{ξ_n} *is a Borel probability measure on* $\mathbb{R}^N$ *for* $\xi_n > 0$, $(\xi_n)_{n\in\mathbb{N}}$ *is bounded. Assume for all* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^N \alpha_k = m$, *we have that*

$$\int_{\mathbb{R}^N} \left(\left(\prod_{k=1}^N |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r \right) d\mu_{\xi_n}(s) < \infty. \tag{5.38}$$

For $\tilde{j} = 1, \ldots, m$, *and* $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_k \in \mathbb{Z}^+$, $k = 1, \ldots, N$, $|\alpha| := \sum_{k=1}^N \alpha_k = \tilde{j}$, *call*

$$c_{\alpha,n,\tilde{j}} := \int_{\mathbb{R}^N} \prod_{k=1}^N s_k^{\alpha_k} d\mu_{\xi_n}(s). \tag{5.39}$$

Then

$$\left\| \left(\theta_{r,n}^{[m]}(f; x) \right)_{\gamma} - f_{\gamma}(x) - \sum_{\tilde{j}=1}^{m} \delta_{\tilde{j},r}^{[m]} \left(\sum_{|\alpha|=\tilde{j}} \frac{c_{\alpha,n,\tilde{j}} f_{(\gamma+\alpha)}(x)}{\prod_{k=1}^N \alpha_k!} \right) \right\|_{1,x} \tag{5.40}$$

$$\leq \sum_{|\alpha|=m} \left(\frac{1}{\prod_{k=1}^N \alpha_k!} \right) \left(\omega_r \left(f_{1,\gamma+\alpha}, \xi_n \right)_1 + \omega_r \left(f_{2,\gamma+\alpha}, \xi_n \right)_1 \right) \cdot \int_{\mathbb{R}^N} \left(\prod_{k=1}^N |s_k|^{\alpha_k} \right) \left(1 + \frac{\|s\|_2}{\xi_n} \right)^r d\mu_{\xi_n}(s).$$

Proof. By Theorem 15 of [1]. ∎

5.3 Applications

Let all entities as in Sect. 5.2. We define the following specific operators for $f\colon \mathbb{R}^N \to \mathbb{C}$.

(a) The general multivariate Picard singular integral operators:

$$P_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \frac{1}{(2\xi_n)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{5.41}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N} |s_i|\right)}{\xi_n}} ds_1 \ldots ds_N.$$

(b) The general multivariate Gauss–Weierstrass singular integral operators:

$$W_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \frac{1}{\left(\sqrt{\pi \xi_n}\right)^N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{5.42}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, x_2 + s_2 j, \ldots, x_N + s_N j)\, e^{-\frac{\left(\sum_{i=1}^{N} s_i^2\right)}{\xi_n}} ds_1 \ldots ds_N.$$

(c) The general multivariate Poisson–Cauchy singular integral operators:

$$U_{r,n}^{[m]}(f; x_1, \ldots, x_N) := W_n^N \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{5.43}$$

$$\int_{\mathbb{R}^N} f(x_1 + s_1 j, \ldots, x_N + s_N j) \prod_{i=1}^{N} \frac{1}{\left(s_i^{2\alpha} + \xi_n^{2\alpha}\right)^\beta} ds_1 \ldots ds_N,$$

with $\alpha \in \mathbb{N}$, $\beta > \frac{1}{2\alpha}$, and

$$W_n := \frac{\Gamma(\beta)\, \alpha \xi_n^{2\alpha\beta - 1}}{\Gamma\left(\frac{1}{2\alpha}\right) \Gamma\left(\beta - \frac{1}{2\alpha}\right)}. \tag{5.44}$$

(d) The general multivariate trigonometric singular integral operators:

$$T_{r,n}^{[m]}(f; x_1, \ldots, x_N) := \lambda_n^{-N} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \cdot \tag{5.45}$$

$$\int_{\mathbb{R}^N} f\left(x_1 + s_1 j, \ldots, x_N + s_N j\right) \prod_{i=1}^{N} \left(\frac{\sin\left(\frac{s_i}{\xi_n}\right)}{s_i} \right)^{2\beta} \mathrm{d}s_1 \ldots \mathrm{d}s_N,$$

where $\beta \in \mathbb{N}$, and

$$\lambda_n := 2\xi_n^{1-2\beta} \pi \left(-1\right)^{\beta} \beta \sum_{k=1}^{\beta} \left(-1\right)^k \frac{k^{2\beta-1}}{(\beta - k)!\,(\beta + k)!}. \tag{5.46}$$

One can apply the results of this chapter to the operators $P_{r,n}^{[m]}$, $W_{r,n}^{[m]}$, $U_{r,n}^{[m]}$, $T_{r,n}^{[m]}$ (special cases of $\theta_{r,n}^{[m]}$) and derive interesting results.

5.4 Conclusion

Our approximation results here imply important convergence properties of operators $\theta_{r,n}^{[m]}$ to the unit operator.

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Chapter 6
Approximation of Functions of Two Variables via Almost Convergence of Double Sequences

The idea of almost convergence for double sequences was introduced by Moricz and Rhoades [6]. In this chapter, we use this concept to prove a Korovkin-type approximation theorem for functions of two variables and we give an example. Furthermore, we present the consequences of the main theorem. This chapter is based on [1].

6.1 Introduction and Preliminaries

Let c and l_∞ denote the spaces of all convergent and bounded sequences, respectively, and note that $c \subset l_\infty$. In the theory of sequence spaces, a beautiful application of the well-known Hahn–Banach Extension Theorem gave rise to the concept of the Banach limit. That is, the lim functional defined on c can be extended to the whole of l_∞ and this extended functional is known as the Banach limit [2], which was used by Lorentz [5] to define a new type of convergence, known as the almost convergence.

A double sequence $x = (x_{\mathrm{jk}})$ of real or complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{j,k} |x_{\mathrm{jk}}| < \infty$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$.

A double sequence $x = (x_{\mathrm{jk}})$ is said to *converge to the limit L in Pringsheim's sense* (shortly, *p-convergent to L*) [10] if for every $\varepsilon > 0$ there exists an integer N such that $|x_{\mathrm{jk}} - L| < \varepsilon$ whenever $j, k > N$. In this case L is called the p-limit of x. If in addition $x \in \mathcal{M}_u$, then x is said to be *boundedly convergent to L in Pringsheim's sense* (shortly, *bp-convergent to L*).

G.A. Anastassiou, *Approximation by Multivariate Singular Integrals*, SpringerBriefs in Mathematics, DOI 10.1007/978-1-4614-0589-4_6,

Let Ω denote the vector space of all double sequences with the vector space operations defined coordinatewise. Vector subspaces of Ω are called *double sequence spaces*. In addition to the above-mentioned double sequence spaces, we consider the double sequence space

$$\mathcal{L}_u := \left\{ x \in \Omega \mid \|x\|_1 := \sum_{j,k} |x_{\mathrm{jk}}| < \infty \right\}$$

of all absolutely summable double sequences.

All considered double sequence spaces are supposed to contain

$$\Phi := \text{span}\,\{\mathrm{e}^{\mathrm{jk}} \mid j, k \in\},$$

where

$$\mathbf{e}^{\mathrm{jk}}_{\mathrm{il}} = \begin{cases} 1 \ ; \ \text{if} \ (j,k) = (i,\ell), \\ 0 \ ; \ \text{otherwise.} \end{cases}$$

We denote the pointwise sums $\sum_{j,k} \mathrm{e}^{\mathrm{jk}}$, $\sum_j \mathrm{e}^{\mathrm{jk}}$ $(k \in)$, and $\sum_k \mathrm{e}^{\mathrm{jk}}$ $(j \in)$ by $\mathbf{e}$, $\mathrm{e}^{\mathbf{k}}$ and $\mathbf{e_j}$ respectively.

The idea of almost convergence for double sequences was introduced and studied by Moricz and Rhoades [6].

A double sequence $x = (x_{\mathrm{jk}})$ of real numbers is said to be *almost convergent* to a limit L if

$$\lim_{p,q\to\infty} \sup_{m,n>0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{\mathrm{jk}} - L \right| = 0 \qquad (*)$$

In this case, L is called the F_2-limit of x and we shall denote by F_2 the space of all almost convergent double sequences.

Note that a convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded.

Example 6.1. The double sequence $z = (z_{\mathrm{mn}})$ defined by

$$z_{\mathrm{mn}} = \begin{cases} 1 \ \text{if}\ m = n \ \text{odd}, \\ -1 \ \text{if}\ m = n \ \text{even}, \\ 0 \ (m \neq n); \end{cases} \qquad (6.1)$$

is almost convergent to zero but not convergent.

For recent developments on almost convergent double sequences and matrix transformations, we refer to [7–9, 11].

If $m = n = 1$ in $(*)$, then we get $(C, 1, 1)$-convergence, and in this case we write $x_{\mathrm{jk}} \to \ell(C, 1, 1)$; where $\ell = (C, 1, 1)$-$\lim x$.

Let $C[a,b]$ be the space of all functions f continuous on $[a,b]$. We know that $C[a,b]$ is a Banach space with norm

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|,\ f \in C[a,b].$$

The classical Korovkin approximation theorem states as follows [4]:

Let (T_n) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Then $\lim_n \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[a,b]$ if and only if $\lim_n \|T_n(f_i,x) - f_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Quite recently, such type of approximation theorems are proved in [3] for functions of two variables by using statistical convergence. In this chapter, we use the notion of almost convergence to prove approximation theorems for functions of two variables.

6.2 Korovkin-Type Approximation Theorem

The following is the F_2-version of the classical Korovkin approximation theorem followed by an example to show its importance.

Let $C(I^2)$ be the space of all two dimensional continuous functions on $I \times I$, where $I = [a,b]$. Suppose that $T_{m,n} : C(I^2) \to C(I^2)$. We write $T_{m,n}(f;x,y)$ for $T_{m,n}(f(s,t);x,y)$; and we say that T is a positive operator if $T(f;x,y) \geq 0$ for all $f(x,y) \geq 0$.

Theorem 6.2. *Let $(T_{j,k})$ be a double sequence of positive linear operators from $C(I^2)$ into $C(I^2)$ and $D_{m,n,p,q}(f;x,y) = \frac{1}{pq}\sum_{j=m}^{m+p-1}\sum_{k=n}^{n+q-1} T_{j,k}(f;x,y)$. Then for all $f \in C(I^2)$*

$$F_2\text{-}\lim_{j,k\to\infty}\left\| T_{j,k}(f;x,y) - f(x,y)\right\|_\infty = 0,\ i.e.$$

$$\lim_{p,q\to\infty}\left\| D_{m,n,p,q}(f;x,y) - f(x,y)\right\|_\infty = 0,\ \textit{uniformly in } m,n. \tag{6.2}$$

if and only if

$$\lim_{p,q\to\infty}\left\| D_{m,n,p,q}(1;x,y) - 1\right\|_\infty = 0 \textit{ uniformly in } m,n, \tag{6.3}$$

$$\lim_{p,q\to\infty}\left\| D_{m,n,p,q}(s;x,y) - x\right\|_\infty = 0 \textit{ uniformly in } m,n, \tag{6.4}$$

$$\lim_{p,q\to\infty}\left\| D_{m,n,p,q}(t;x,y)-y\right\|_\infty = 0 \text{ uniformly in } m,n. \quad (6.5)$$

$$\lim_{p,q\to\infty}\left\| 4D_{m,n,p,q}(s^2+t^2;x,y)-(x^2+y^2)\right\|_\infty = 0 \text{ uniformly in } m,n. \quad (6.6)$$

Proof. Since each $1, x, y, x^2+y^2$ belongs to $C(I^2)$, conditions (6.3)–(6.6) follow immediately from (6.2). By the continuity of f on I^2, we can write $|f(x,y)| \leq M, \ -\infty < x, y < \infty$, where $M = \|f\|_\infty$. Therefore,

$$|f(s,t)-f(x,y)| \leq 2M, \ -\infty < s,t,x,y < \infty. \quad (6.7)$$

Also, since $f \in C(I^2)$, for every $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(s,t)-f(x,y)| < \epsilon, \ \ \forall\, |s-x| < \delta \text{ and } |t-y| < \delta. \quad (6.8)$$

Using (6.7), (6.8), putting $\psi_1 = \psi_1(s,x) = (s-x)^2$ and $\psi_2 = \psi_2(t,y) = (t-y)^2$, we get

$$|f(s,t)-f(x,y)| < \epsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2), \ \ \forall\, |s-x| < \delta \text{ and } |t-y| < \delta.$$

This is,

$$-\epsilon - \frac{2M}{\delta^2}(\psi_1+\psi_2) < f(s,t)-f(x,y) < \epsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2).$$

Now, operating $T_{j,k}(1;x,y)$ to this inequality since $T_{j,k}(f;x,y)$ is monotone and linear. We obtain

$$T_{j,k}(1;x,y)\left(-\epsilon - \frac{2M}{\delta^2}(\psi_1+\psi_2)\right) < T_{j,k}(1;x,y)(f(s,t)-f(x,y))$$
$$< T_{j,k}(1;x,y)\left(\epsilon + \frac{2M}{\delta^2}(\psi_1+\psi_2)\right).$$

Note that x and y are fixed and so $f(x,y)$ is constant number. Therefore

$$-\epsilon T_{j,k}(1;x,y) - \frac{2M}{\delta^2}T_{j,k}(\psi_1+\psi_2;x,y) < T_{j,k}(f;x,y) - f(x,y)T_{j,k}(1;x,y)$$
$$< \epsilon T_{j,k}(1;x,y) + \frac{2M}{\delta^2}T_{j,k}(\psi_1+\psi_2;x,y). \quad (6.9)$$

But

$$\begin{aligned}T_{j,k}(f;x,y)-f(x,y)&=T_{j,k}(f;x,y)-f(x,y)T_{j,k}(1;x,y)\\&\quad+f(x,y)T_{j,k}(1;x,y)-f(x,y)\\&=[T_{j,k}(f;x,y)-f(x,y)T_{j,k}(1;x,y)]\\&\quad+f(x,y)[T_{j,k}(1;x,y)-1].\end{aligned}\tag{6.10}$$

Using (6.9) and (6.10), we have

$$\begin{aligned}T_{j,k}(f;x,y)-f(x,y)<\epsilon T_{j,k}(1;x,y)+\frac{2M}{\delta^2}T_{j,k}(\psi_1+\psi_2;x,y)\\+f(x,y)(T_{j,k}(1;x,y)-1).\end{aligned}\tag{6.11}$$

Now

$$\begin{aligned}T_{j,k}(\psi_1+\psi_2;x,y)&=T_{j,k}((s-x)^2+(t-y)^2;x,y)\\&=T_{j,k}(s^2-2sx+x^2+t^2-2ty+y^2;x,y)\\&=T_{j,k}(s^2+t^2;x,y)-2xT_{j,k}(s;x,y)-2yT_{j,k}(t;x,y)\\&\quad+(x^2+y^2)T_{j,k}(1;x,y)\\&=[T_{j,k}(s^2+t^2;x,y)-(x^2+y^2)]-2x[T_{j,k}(s;x,y)-x]\\&\quad-2y[T_{j,k}(t;x,y)-y]+(x^2+y^2)[T_{j,k}(1;x,y)-1].\end{aligned}$$

Using (6.11), we obtain

$$\begin{aligned}T_{j,k}(f;x,y)-f(x,y)&<\epsilon T_{j,k}(1;x,y)+\frac{2M}{\delta^2}\{[T_{j,k}(s^2+t^2;x,y)-(x^2+y^2)]\\&\quad-2x[T_{j,k}(s;x,y)-x]-2y[T_{j,k}(t;x,y)-y]\\&\quad+(x^2+y^2)[T_{j,k}(1;x,y)-1]\}\\&\quad+f(x,y)(T_{j,k}(1;x,y)-1)\\&=\epsilon[T_{j,k}(1;x,y)-1]+\epsilon+\frac{2M}{\delta^2}\{[T_{j,k}(s^2+t^2;x,y)\\&\quad-(x^2+y^2)]-2x[T_{j,k}(s;x,y)-x]\\&\quad-2y[T_{j,k}(t;x,y)-y]+(x^2+y^2)[T_{j,k}(1;x,y)-1]\}\\&\quad+f(x,y)(T_{j,k}(1;x,y)-1).\end{aligned}$$

Since ϵ is arbitrary, we can write

$$
\begin{aligned}
T_{j,k}(f;x,y) - f(x,y) \le \epsilon[T_{j,k}(1;x,y)-1] + \frac{2M}{\delta^2}\{[T_{j,k}(s^2+t^2;x,y) \\
-(x^2+y^2)] - 2x[T_{j,k}(s;x,y)-x] \\
-2y[T_{j,k}(t;x,y)-y] + (x^2+y^2)[T_{j,k}(1;x,y)-1]\} \\
+ f(x,y)(T_{j,k}(1;x,y)-1).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
D_{m,n,p,q}(f;x,y) - f(x,y) \le \epsilon[D_{m,n,p,q}(1;x,y)-1] \\
+\frac{2M}{\delta^2}\big\{[D_{m,n,p,q}(s^2+t^2;x,y)-(x^2+y^2)] \\
-2x[D_{m,n,p,q}(s;x,y)-x] \\
-2y[D_{m,n,p,q}(t;x,y)-y] \\
+(x^2+y^2)[D_{m,n,p,q}(1;x,y)-1]\big\} \\
+f(x,y)(D_{m,n,p,q}(1;x,y)-1)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\left\|D_{m,n,p,q}(f;x,y)-f(x,y)\right\|_\infty \le \left(\epsilon + \frac{2M(a^2+b^2)}{\delta^2} + M\right) \\
\times \left\|D_{m,n,p,q}(1;x,y)-1\right\|_\infty \\
-\frac{4Ma}{\delta^2}\left\|D_{m,n,p,q}(s;x,t)-x\right\|_\infty \\
-\frac{4Mb}{\delta^2}\left\|D_{m,n,p,q}(t;x,y)-y\right\|_\infty \\
+\frac{2M}{\delta^2}\left\|D_{m,n,p,q}(s^2+t^2;x,y)-(x^2+y^2)\right\|_\infty.
\end{aligned}
$$

Letting $p, q \to \infty$ and using (6.3), (6.4), (6.5), and (6.6), we get

$$\lim_{p,q\to\infty} \left\| D_{m,n,p,q}(f; x, y) - f(x, y) \right\|_{\infty} = 0, \text{ uniformly in } m, n.$$

This completes the proof of the theorem. □

In the following, we give an example of a double sequence of positive linear operators satisfying the conditions of Theorem 6.2 but does not satisfy the conditions of the Korovkin theorem.

Example 6.3. Consider the sequence of classical Bernstein polynomials of two variables

$$B_{m,n}(f; x, y) := \sum_{j=0}^{m} \sum_{k=0}^{n} f\left(\frac{j}{m}, \frac{k}{n}\right) \binom{m}{j} \binom{n}{k} x^j (1-x)^{m-j}$$

$$\times\, y^k (1-y)^{n-k}; 0 \leq x, y \leq 1.$$

Let $P_{m,n} : C(I^2) \to C(I^2)$ be defined by

$$P_{m,n}(f; x, y) = (1 + z_{\mathrm{mn}}) B_{m,n}(f; x, y),$$

where (z_{mn}) is a double sequence defined as above. Then

$$B_{m,n}(1; x, y) = 1,$$

$$B_{m,n}(s; x, y) = x,$$

$$B_{m,n}(t; x, y) = y,$$

$$B_{m,n}(s^2 + t^2; x, y) = x^2 + y^2 + \frac{x - x^2}{m} + \frac{y - y^2}{n},$$

and a double sequence $(P_{m,n})$ satisfies the conditions (6.3), (6.4), (6.5) and (6.6). Hence, we have

$$F_2\text{-}\lim_{m,n\to\infty} \| P_{m,n}(f; x, y) - f(x, y) \|_{\infty} = 0.$$

On the other hand, we get $P_{m,n}(f; 0, 0) = (1 + z_{\mathrm{mn}}) f(0, 0)$, since $B_{m,n}(f; 0, 0) = f(0, 0)$, and hence

$$\| P_{m,n}(f; x, y) - f(x, y) \|_{\infty} \geq | P_{m,n}(f; 0, 0) - f(0, 0) | = z_{\mathrm{mn}} | f(0, 0) |.$$

We see that $(P_{m,n})$ does not satisfy the classical Korovkin theorem, since $\lim\limits_{m,n\to\infty} z_{\mathrm{mn}}$ does not exist.

6.3 Some Consequences

Now we present here some consequences of Theorem 6.2.

Theorem 6.4. *Let $(T_{m,n})$ be a double sequence of positive linear operators on $C(I^2)$ such that*

$$\lim_{m,n} \|T_{m+1,n+1} - T_{m,n+1} - T_{m+1,n} + T_{m,n}\| = 0. \tag{6.12}$$

If

$$F_2\text{-}\lim_{m,n} \|T_{m,n}(t_\nu; x, y) - t_\nu\|_\infty = 0 \quad (\nu = 0, 1, 2, 3), \tag{6.13}$$

where $t_0(x, y) = 1$, $t_1(x, y) = x$, $t_2(x, y) = y$ and $t_3(x, y) = x^2 + y^2$. Then for any function $f \in C(I^2)$, we have

$$\lim_{m,n} \|T_{m,n}(f; x, y) - f(x, y)\|_\infty = 0. \tag{6.14}$$

Proof. From Theorem 6.2, we have that if (6.13) holds then

$$\lim_{p,q} \|D_{m,n,p,q}(f; x, y) - f(x, y)\|_\infty = 0, \text{uniformly in } m, n. \tag{6.15}$$

We have the following inequality

$$\begin{aligned}
&\|T_{m,n}(f; x, y) - f(x, y)\|_\infty \\
&\quad \le \|D_{m,n,p,q}(f; x, y) - f(x, y)\|_\infty \\
&\qquad + \frac{1}{pq} \sum_{j=m+1}^{m+p-1} \sum_{k=n+1}^{n+q-1} \left(\sum_{\alpha=m+1}^{j} \sum_{\beta=n+1}^{k} \|T_{\alpha,\beta} - T_{\alpha-1,\beta} - T_{\alpha,\beta-1} + T_{\alpha-1,\beta-1}\| \right) \\
&\quad \le \|D_{m,n,p,q}(f; x, y) - f(x, y)\|_\infty \\
&\qquad + \frac{p-1}{2} \frac{q-1}{2} \left\{ \sup_{j \ge m, k \ge n} \|T_{j,k} - T_{j-1,k} - T_{j,k-1} + T_{j-1,k-1}\| \right\}.
\end{aligned} \tag{6.16}$$

Hence using (6.12) and (6.15), we get (6.14).

This completes the proof of the theorem. □

We know that double almost convergence implies $(C, 1, 1)$ convergence. This motivates us to further generalize our main result by weakening the hypothesis or to add some condition to get more general result.

Theorem 6.5. *Let $(T_{m,n})$ be a double sequence of positive linear operators on $C(I^2)$ such that*

$$(C, 1, 1) - \lim_{m,n} \|T_{m,n}(t_\nu, x) - t_\nu\|_\infty = 0 \quad (\nu = 0, 1, 2, 3), \tag{6.17}$$

and

$$\lim_{p,q}\left\{\sup_{m\geq p,n\geq q}\frac{mn}{pq}\left\|\sigma_{m+p-1,n+q-1}(f;x,y)-\sigma_{m-1,n-1}(f;x,y)\right\|_{\infty}\right\}=0, \quad (6.18)$$

where

$$\sigma_{m,n}(f;x,y)=\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}T_{j,k}(f;x,y).$$

Then for any function $f\in C(I^2)$*, we have*

$$F_2\text{-}\lim_{m,n\to\infty}\left\|T_{m,n}(f;x,y)-f(x,y)\right\|_{\infty}=0.$$

Proof. For $m\geq p\geq 1; n\geq q\geq 1$, it is easy to show that

$$D_{m,n,p,q}(f;x,y)=\sigma_{m+p-1,n+q-1}(f;x,y)+\frac{mn}{pq}(\sigma_{m+p-1,n+q-1}(f;x,y)$$
$$-\sigma_{m-1,n-1}(f;x,y)),$$

which implies

$$\sup_{m\geq p,n\geq q}\left\|D_{m,n,p,q}(f;x,y)-\sigma_{m+p-1,n+q-1}(f;x,y)\right\|_{\infty}$$
$$=\sup_{m\geq p,n\geq q}\frac{mn}{pq}\left\|\sigma_{m+p-1,n+q-1}(f;x,y)-\sigma_{m-1,n-1}(f;x,y))\right\|_{\infty}. \quad (6.19)$$

Also by Theorem 6.2, Condition (6.17) implies that

$$(C,1,1)\text{-}\lim_{m,n\to\infty}\left\|T_{m,n}(f;x,y)-f(x,y)\right\|_{\infty}=0. \quad (6.20)$$

Using (6.17)–(6.20) and the fact that almost convergence implies $(C,1,1)$ convergence, we get the desired result.

This completes the proof of the theorem. □

Theorem 6.6. *Let* $(T_{m,n})$ *be a double sequence of positive linear operators on* $C(I^2)$ *such that*

$$\lim_{m,n}\sup_{s,t}\frac{1}{mn}\sum_{j=s}^{s+m-1}\sum_{k=t}^{t+n-1}\|T_{m,n}-T_{j,k}\|=0.$$

If

$$F_2\text{-}\lim_{m,n} \|T_{m,n}(t_\nu, x) - t_\nu\|_\infty = 0 \quad (\nu = 0, 1, 2, 3). \tag{6.21}$$

Then for any function $f \in C(I^2)$, we have

$$\lim_{m,n} \|T_{m,n}(f; x, y) - f(x, y)\|_\infty = 0. \tag{6.22}$$

Proof. From Theorem 6.2, we have that if (6.21) holds then

$$F_2\text{-}\lim_{m,n} \|T_{m,n}(f; x, y) - f(x, y)\|_\infty = 0,$$

which is equivalent to

$$\lim_{m,n} \sup_{s,t} \|D_{s,t,m,n}(f; x, y) - f(x, y)\|_\infty = 0. \tag{6.23}$$

Now

$$\begin{aligned} T_{m,n} - D_{s,t,m,n} &= T_{m,n} - \frac{1}{mn} \sum_{j=s}^{s+m-1} \sum_{k=t}^{t+n-1} T_{j,k} \\ &= \frac{1}{mn} \sum_{j=s}^{s+m-1} \sum_{k=t}^{t+n-1} (T_{m,n} - T_{j,k}). \end{aligned}$$

Therefore,

$$\sup_{s,t} \|T_{m,n} - D_{s,t,m,n}\|_\infty \le \sup_{s,t} \frac{1}{mn} \sum_{j=s}^{s+m-1} \sum_{k=t}^{t+n-1} \|T_{m,n} - T_{j,k}\|.$$

Now, by using the hypothesis we get

$$\lim_{m,n} \sup_{s,t} \|T_{m,n}(f; x, y) - D_{s,t,m,n}(f; x, y)\|_\infty = 0. \tag{6.24}$$

By the triangle inequality, we have

$$\begin{aligned} \|T_{m,n}(f; x, y) - f(x, y)\|_\infty &\le \|T_{m,n}(f; x, y) - D_{s,t,m,n}(f; x, y)\|_\infty \\ &\quad + \|D_{s,t,m,n}(f; x, y) - f(x, y)\|_\infty, \end{aligned}$$

and hence from (6.23) and (6.24), we get

$$\lim_{m,n} \|T_{m,n}(f; x, y) - f(x, y)\|_\infty = 0,$$

that is (6.22) holds.

This completes the proof of the theorem. □

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